On the Pointwise Convergence of Fourier Series

Stanley Hong

MATH 425b Final Project

Abstract.
In this project I will discuss the properties of pointwise convergence of Fourier series. In our MATH 425b lecture, we learned that by defining

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,\mathrm{d}\theta, \;\; S_N f = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta},$$

we can approximate any $f \in L^2_{\mathcal{R}}[-\pi,\pi]$ with Fourier series $S_N f$ with *mean-square convergence*:

$$\lim_{N\to\infty} \|S_n f - f\|_2 = 0.$$

However, this does *not* give the full picture, as we still cannot approximate f *pointwise*, not to say *uniformly*. By considering the N-th partial sum of the Fourier series

$$S_n f = \frac{1}{2\pi}\int_{\mathbb{R}} D_N(x-y)f(y)\,\mathrm{d}y = \frac{1}{2\pi}D_N * f$$

using a family of functions D_N called the *Dirichlet kernels*, we recognize some drawbacks of the Dirichlet kernel and therefore the Fourier series. Namely, The Dirichlet kernel is *not* an approximation identity (section 1), so the limit of its convolution with f does not necessarily tend to f pointwise, except when f is Lipschitz (section 3). Particularly we will prove the existence of a function such that $S_n f(x) \to \infty$ at some x, contrary to the boundedness of f (section 2).
Then we will move on to discuss *good kernels*, namely the *Fejér kernel*

$$F_N(x) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(x),$$

the result of *Cesàro summation* on the Dirichlet kernel (section 4). An approximation identity, the Fejér kernel allows for pointwise and even uniform convergence of the uniform series under convolution $F_N * f$ (section 5).
Lastly, we will introduce an interesting phenomenon about Fourier series: the *Gibbs's phenomenon*, which describes the overshooting behavior of Fourier series near jump discontinuities.
Without loss of generality, I will limit my approximated function f as a $L^2_{\mathcal{R}}[-\pi,\pi]$ function: a 2π-periodic Riemann integrable L^2-bounded (in the Riemann sense) function.

Section 1. The Dirichlet Kernel and its Problems

We begin the section by introducing the Dirichlet Kernel.

Definition 1. The N-th **Dirichlet kernel** $D_N(x) : [-\pi, \pi] \to \mathbb{C}$ is defined by the trigonometric polynomial

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = e^{-iNx} + \cdots + e^{-ix} + 1 + e^{ix} + \cdots + e^{iNx}.$$

The Dirichlet kernel itself has Fourier coefficients $\hat{f}(n) = 1$ for all $|n| \leqslant N$, and $\hat{f}(n) = 0$ otherwise. Recalling the N-th partial sum of Fourier series of f:

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \hat{f}(-N)e^{-iNx} + \cdots + \hat{f}(-1)e^{-ix} + \hat{f}(0) + \hat{f}(1)e^{ix} + \cdots + \hat{f}(N)e^{iNx},$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} \, \mathrm{d}\theta,$$

We see similarities between D_N and $S_N f$, but we are still unsure about their exact relation. The following proposition establishes the relation.

Proposition 2.

$$S_N(f)(x) = (f * D_N)(x).$$

Proof. First we expand $S_N(f)(x)$ to obtain

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = \sum_{n=-N}^{N} \frac{1}{2\pi} e^{inx} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} \, \mathrm{d}\theta.$$

For each $|n| \leqslant N$, notice that

$$\frac{1}{2\pi} e^{inx} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} \, \mathrm{d}\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{inx-in\theta} \, \mathrm{d}\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{in(x-\theta)} \, \mathrm{d}\theta.$$

Now denote $S_n(f)(x) := \hat{f}(n)e^{inx}$, $|n| \leqslant N$ to denote the "n-th term" of the partial sum. Similarly, denote $D_n(x) = e^{inx}$ as the "n-th term" of the Dirichlet kernel. We claim that $S_n(f)(x) \propto D_n * f(x)$. Indeed,

$$S_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{in(x-\theta)} \, \mathrm{d}\theta = (D_n * f)(x).$$

The general case $S_N(f)(x) = (f * D_N)(x)$ follows from the linearity of convolution. □

Remark. It is important to note that we actually "cheated" a bit and used a slightly different notation of convolution. The most standard definition of convolution of $f, g \in \mathcal{R}_{\text{loc}}$ on $\mathbb{R}$ is

$$f * g(x) = \int_{\mathbb{R}} f(y)g(x-y) \, \mathrm{d}y.$$

However, considering 2π-periodic functions $f, g \in \mathcal{R}_{\text{loc}}$, it makes more sense to define

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)g(y) \, \mathrm{d}y.$$

This way the convolution would make more sense as a "weighed average". Merely a scalar multiplication, the algebraic properties of convolution are preserved.

Proposition 3. The N-th Dirichlet exhibits a closed form in

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}.$$

Proof. Denote $\omega := e^{ix}$. First we split the summation for two geometric sequences with common multiple ω:

$$D_N(x) = \sum_{n=-N}^{N} \omega^n = \sum_{n=-N}^{-1} \omega^n + \sum_{n=0}^{N} \omega^n.$$

The first term of negative exponents has first term ω^{-N}, hence the sum of the sequence is

$$\sum_{n=-N}^{-1} \omega^n = \frac{\omega^{-N}(1-\omega^N)}{1-\omega}.$$

Similarly, the second term of nonnegative exponents has first term $\omega^0 = 1$, hence the sum of the sequence is

$$\sum_{n=0}^{N} \omega^N = \frac{1(1-\omega^{N+1})}{1-\omega}.$$

Summing the two parts, we obtain

$$D_N(x) = \frac{\omega^{-N} - \omega^{N+1}}{1-\omega} = \frac{\omega^{-N-0.5} - \omega^{N+0.5}}{\omega^{-0.5} - \omega^{0.5}}.$$

Note that in the second "=" we divided all terms by $\omega^{0.5}$. As $\omega = e^{ix} \in \mathbb{C}$, we can use the fact that $z - \overline{z} = 2\text{Im } z$ to represent the expression as

$$D_N(x) = \frac{-2\text{Im } \omega^{N+0.5}}{-2\text{Im } \omega^{0.5}} = \frac{\text{Im}\left(\text{cis } e^{(N+\frac{1}{2})ix}\right)}{\text{Im}\left(\text{cis } e^{\frac{1}{2}ix}\right)} = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)},$$

as desired. □

Remark. Although D_N involves the use of imaginary numbers $\omega = e^{iNx}$, proposition 3 actually implies that D_N is a function that maps $\mathbb{R}$ to $\mathbb{R}$ - the imaginary parts get cancelled out.

In our 425b lecture, we proved the proposition that the Dirichlet kernel performs well when convolving with the target function f in estimating the L^2 norm of f. The idea is to use triangular inequality to bounded $\|S_N f - f\|_2$ with components that tend to zero under certain conditions using the L^1 approximation theorem, Stone-Weierstraß, and properties of orthogonal projection. Hence we will state the proposition here without proof.

Proposition 4. (L^2 *approximation of Dirichlet kernel*)

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0.$$

Nevertheless, the Dirichlet kernel is *not* a *good kernel*. Primarily, it is *not* an approximate identity.
We know that *if* $\{D_N\}_N$ is indeed an approximate identity, then with $f \in L^2_{\mathcal{R}}[-\pi,\pi]$ bounded, we have that

$$\lim_{N\to\infty} D_N * f(x) = f(x)$$

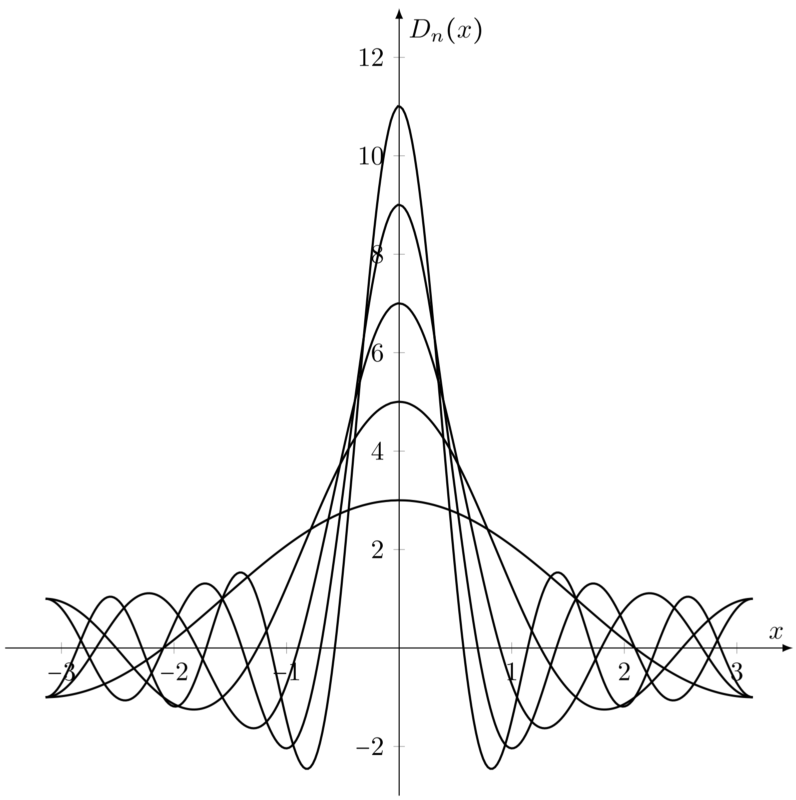
for each x of continuity (of f). Therefore, for $f \in C(a,b)$, we have that

$$\lim_{N\to\infty} D_N * f \rightrightarrows f \text{ on any } [c,d] \subset (a,b).$$

We first show that $\{D_N\}_N$ is not an approximate identity in the following proposition, then in the next section (section 2) we will further prove the existence of a function where the Dirichlet kernel *officially* fails to work.

Proposition 5. The Dirichlet kernel $\{D_N\}_N$ is not an approximate identity.

Proof. We test $\{D_N\}_N$ on the defining properties of an approximate identity, before which we shall have a *look* at an approximate shape of $\{D_N\}_N$.



The above graph displays D_N for $N \in [5]$. It *seems* like although the sequence *does* indeed look like an approximate identity, the increasing amplitude on the negative side may be troublesome as $\|D_N\|_1$ may no longer be uniformly bounded. We **claim** that the Dirichlet kernel fails to satisfy this property.To see, so, first observe that D_N, as the quotient of two odd functions, is even. Therefore we only need to consider $\|D_N\|_{1[0,\pi]} = \int_0^\pi |D_N| \, \mathrm{d}x$. Considering its closed form expansion, we have that

$$\begin{aligned}
\int_0^\pi \left|\frac{\sin(N+\frac{1}{2})x}{\sin(x/2)}\right| \mathrm{d}x &= 2\int_0^{\frac{\pi}{2}} \left|\frac{\sin((2N+1)x)}{\sin x}\right| \mathrm{d}x && \text{(Change of vars: } \frac{x}{2} \mapsto x\text{)} \\
&> 2\int_0^{\frac{\pi}{2}} \frac{|\sin((2N+1)x)|}{x} \mathrm{d}x && \text{(Comparison: } |\sin x| < x \text{ on } x \in [0,\pi/2]\text{)} \\
&= 2\int_0^{(2N+1)\frac{\pi}{2}} \frac{|\sin u|}{u} \mathrm{d}u && \text{(Change of vars: } (2N+1)x \mapsto u\text{)} \\
&> 2\int_0^{N\pi} \frac{|\sin u|}{u} \mathrm{d}u = I && \text{(Truncating integer multiples of } \pi\text{).}
\end{aligned}$$

From the lower bound I we can already see the trend that I is unbounded. Indeed, a few more intermediary steps gives an explicit divergent lower bound that proves the claim. Note that

$$I = \int_0^{n\pi} \frac{|\sin u|}{u} \, \mathrm{d}u = \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u} \, \mathrm{d}u.$$

For $k \in [n-1]_0$, $(k+1)\pi \geqslant u$, hence

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u} \, \mathrm{d}u \geqslant \int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{(k+1)\pi} \, \mathrm{d}u = \frac{1}{(k+1)\pi} \underbrace{\int_{k\pi}^{(k+1)\pi} |\sin u| \, \mathrm{d}u}_{=2} = \frac{2}{(k+1)\pi}.$$

Therefore

$$I = 2\sum_{k=0}^{n-1} \frac{2}{(k+1)\pi} = \frac{4}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1}.$$

The final term returns the n-th partial sum of a *harmonic series*, which trivially diverges. In fact, we can impose a lower bound of $\log n$, which also diverges, to conclude $\|D_N\|_1 > 4\pi^{-1} \log n \to \infty$. □

Remark. The Dirichlet kernel does satisfy the first property of unit signed mass.

Proof. For $N = 0$, $D_0 \equiv 1$, hence its signed mass over $[-\pi, \pi]$ is 2π. We further claim that D_N also has signed mass of 2π. Indeed, consider each D_N as an increment from D_{N-1}:

$$D_N = D_{N-1} + e^{iNx} + e^{-iNx} \Rightarrow \int_{-\pi}^{\pi} D_N \, \mathrm{d}x = \int_{-\pi}^{\pi} D_{N-1} \, \mathrm{d}x + \int_{-\pi}^{\pi} e^{iNx} + e^{-iNx} \, \mathrm{d}x.$$

However, the red part evaluates to zero. Considering $z + \overline{z} = 2\mathrm{Re}\, z$, we have that

$$e^{iNx} + e^{-iNx} = 2\mathrm{Re}\, e^{iNx} = 2\cos Nx,$$

with definite integral $\left[2N^{-1} \sin Nx\right]_{-\pi}^{\pi} = 0$. The argument follows by induction. We can then conclude:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) \, \mathrm{d}x = 1,$$

as desired. □

Section 2. The function where the Dirichlet Kernel fails to work

Earlier we proved that the Dirichlet kernel is *not* an approximate identity. However, as a sufficient condition, disproving the Dirichlet kernel does not provide sufficient grounds to proving its failure to converge to f pointwise. Therefore, we prove that there exists a function f where $S_N f = D_N * f$ fails to converge at some x (say zero for simplicity) by constructing an explicit example.

Proposition 6. There exists a function $f \in L^2_{\mathcal{R}}[-\pi, \pi]$ such that the Fourier series $S_n f$ diverges at zero.

Proof. We separate the question into smaller parts.

Part 1. The sawtooth function

Problem: Stein Ex. 2.8. The **sawtooth function** $f : (-\pi, \pi) \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & x \in (-\pi, 0); \\ 0, & x = 0; \\ \frac{\pi}{2} - \frac{x}{2}, & x \in (0, \pi). \end{cases}$$

Its affiliated Fourier series is

$$\hat{f}(x) = \frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}.$$

Considering the Fourier coefficients, if $\hat{f}(0)$ is trivially zero as $\int_{-\pi}^{\pi} f(\theta)\, \mathrm{d}\theta = 0$. If $n \neq 0$, we have that

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, \mathrm{d}\theta && \text{(Definition of Fourier series)} \\
&= \frac{1}{2\pi} \left[\int_0^{\pi} (\frac{\pi}{2} - \frac{\theta}{2}) e^{-in\theta}\, \mathrm{d}\theta - \int_{-\pi}^{0} (-\frac{\pi}{2} - \frac{\theta}{2}) e^{-in\theta}\, \mathrm{d}\theta \right] && \text{(Definition of Sawtooth)} \\
&= \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} -\frac{\theta}{2} e^{-in\theta}\, \mathrm{d}\theta + \left(\int_0^{\pi} \frac{\pi}{2} e^{-in\theta}\, \mathrm{d}\theta + \int_{-\pi}^{0} -\frac{\pi}{2} e^{-in\theta}\, \mathrm{d}\theta \right) \right] && \text{(Algebra inside integral)} \\
&= \frac{1}{2\pi} (I_1 + I_2).
\end{aligned}$$

Evaluating the integrals on their own, we have, for I_1,

$$\begin{aligned}
I_1 &= \int_{-\pi}^{\pi} -\frac{\theta}{2}\, \mathrm{d}e^{-in\theta} \\
&= \left[\frac{\theta}{2in} e^{-in\theta} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{2} e^{-in\theta}\, \mathrm{d}\theta \\
&= \frac{\pi}{2in} e^{-in\pi} - \frac{\pi}{2in} e^{in\pi} + \left[\frac{1}{2in} e^{-in\theta} \right]_{-\pi}^{\pi} \\
&= \frac{\pi}{2in} 2\cos n\pi - \frac{1}{2in} 2\sin n\pi = \frac{\pi}{in} \cos n\pi.
\end{aligned}$$

Then for I_2,

$$\begin{aligned}
I_2 &= \left[-\frac{\pi}{2in} e^{-in\theta} \right]_0^{\pi} + \left[\frac{\pi}{2in} e^{-in\theta} \right]_{-\pi}^{0} \\
&= -\frac{\pi}{2in} e^{-in\pi} + \frac{\pi}{2in} + \frac{\pi}{2in} - \frac{\pi}{2in} e^{in\pi} \\
&= \frac{\pi}{in} - \frac{\pi}{2in} 2\cos n\pi = \frac{\pi}{in} - \frac{\pi}{in} \cos n\pi.
\end{aligned}$$

We then conclude with that $\hat{f}(n) = \frac{1}{2\pi} \cdot \frac{\pi}{in} = \frac{1}{2ni}$, as desired.

The sawtooth function allows us to observe that the series converges for every x, including the origin. Note that the sawtooth function sees a jump discontinuity at zero, where

$$\lim_{x \to 0^-} f(x) = -\frac{\pi}{2}, \quad f(0) = 0, \quad \lim_{x \to 0^+} f(x) = +\frac{\pi}{2}.$$

When the series is evaluated at zero, its value (zero) is the average of the values of $f(x)$ as x approaches the origin from the left and the right - a symmetry. This allows for an opportunity to *break* it.

Part 2. Symmetry-breaking

We can transform the sawtooth function as we wish. Particularly, consider $f : (-\pi, \pi) \to \mathbb{C}$:

$$f(\theta) = \begin{cases} -i\pi - i\theta, & x \in (-\pi, 0); \\ 0, & x = 0; \\ i\pi - i\theta, & x \in (0, \pi). \end{cases}$$

From Stein 2.8, we know that

$$f(\theta) \propto \sum_{n \neq 0} \frac{e^{in\theta}}{n},$$

and the symmetry is preserved through addition of $+n$ and $-n$ to the series for each n. We can then break the symmetry to obtain the "lower half" of the series:

$$\tilde{f}(\theta) = \sum_{n=-\infty}^{-1} \frac{e^{in\theta}}{n}.$$

Accordingly, we define

$$f_N(\theta) := \sum_{1 \leqslant |n| \leqslant N} \frac{e^{in\theta}}{n}, \quad \tilde{f}_N(\theta) = \sum_{-N \leqslant n \leqslant -1} \frac{e^{in\theta}}{n}.$$

We claim that $\left|\tilde{f}_N(0)\right| \geqslant c \log N$, a trivial property of the harmonic series. Additionally, we claim that $f_N(\theta)$ is uniformly bounded in N and θ using the following lemma.

Lemma. Suppose that the *Abel means* $A_r = \sum_{n=1}^{\infty} r^n c_n$ of the series $\sum_{n=1}^{\infty} c_n$ are bounded as $r \to 1^-$. If $c_n = O(n^{-1})$, then the partial sums $S_N = \sum_{n=1}^{N} c_n$ are bounded.

Proof of lemma. We wish to estimate the difference

$$S_N - A_r = \sum_{n=1}^{N} (c_n - r^n c_n) - \sum_{n=N+1}^{\infty} r^n c_n.$$

As $r \to 1^-$, we can let $r = 1 - N^{-1}$. Additionally, choose M such that $|c_n| \leqslant M n^{-1}$ ($c_n = O(n^{-1})$.) Then

$$\begin{aligned} |S_N - A_r| &\leqslant \sum_{n=1}^{N} |c_n| (1 - r^n) + \sum_{n=N+1}^{\infty} r^n |c_n| && \text{(Triangle Inequality)} \\ &\leqslant M \sum_{n=1}^{N} n^{-1}(1 - r^n) + M \sum_{n=N+1}^{\infty} r^n n^{-1} && \text{(}|c_n| \text{ bounded above)} \\ &\leqslant M \sum_{n=1}^{N} (1 - r) + M N^{-1} \sum_{n=N+1}^{\infty} r^n && (n^{-1} \leqslant N^{-1};\ (1 - r^n) \leqslant n(1 - r)) \\ &\leqslant M N (1 - r) + M N^{-1} (1 - r)^{-1} = 2M. && \text{(Geometric sum)} \end{aligned}$$

Hence if $|A_r| \leqslant M$ as well, $|S_N| \leqslant 3M$.

Applying the lemma to the series defining $f_N(\theta)$, consider the series

$$c_n = \frac{e^{in\theta}}{n} + \frac{e^{-in\theta}}{n}$$

for $n \neq 0$. Clearly $c_n = O(1/|n|)$, and the Abel means of the series are

$$A_r(f)(\theta) = \sum_{n=1}^{\infty} r^n c_n = \sum_{n\neq 0} r^{|n|} \frac{e^{in\theta}}{n},$$

which is bounded in both n and θ as $r \in (0,1]$.

Remark. The Abel means $A_r(f)(\theta)$ is actually the convolution between f and the *Poisson kernel*

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

Therefore $S_N(f)(\theta)$ is uniformly bounded in both N and θ, as desired.

Part 3. Construction of function

Now that f_N and $\tilde{f}_N$ are trigonometric polynomials of degree N, we can define through a displacement factor

$$P_N(\theta) = e^{2iN\theta} f_N(\theta), \quad \tilde{P}_N(\theta) = e^{2iN\theta} \tilde{f}_N(\theta).$$

While f_N has non-vanishing Fourier coefficients when $1 \leqslant |n| \leqslant N$, the coefficients of P_N are non-vanishing for $1 \leqslant |n - 2N| \leqslant N$. Considering the M-th partial sums of the series P_N, we have the following lemma.

Lemma.

$$S_M(P_N) = \begin{cases} P_N, & M \geqslant 3N; \\ \tilde{P}_N, & M = 2N; \\ 0, & M < N. \end{cases}$$

Proof of lemma. Because P_N only has non-vanishing terms on $n \in \{N, N+1, \cdots, 2N-1, 2N+1, \cdots, 3N-1, 3N\}$, the partial sum for all $M < N$ takes zero. For $M \in [N, 2N-1] \cap \mathbb{N}$, S_M truncates the terms of P_N starting from N, until $M = 2N$ where the negative terms of f_N are fully summed. Therefore $S_{2N}(P_N) = \tilde{P}_N$. Similarly, for $M \in [2M+1, 3N] \cap \mathbb{N}$, S_M truncates the terms of P_N starting from $2N+1$, until $M = 3N$ where the positive terms of f_N are also fully summed. For $M > 3N$ the terms of P_N fully vanishes hence $S_M(P_N) = P_N$ thereafter. Hence when $M = 2N$, the summation operator S_M breaks the symmetry of P_N.

Lastly, we construct a convergent series of $\sum \alpha_k$ and a sequence of integers $\{N_k\}_k$, increases rapidly such that

- $N_{k+1} > 3N_k$,
- $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

Taking $\alpha_k = k^{-2}$ and $N_k = 3^{2^k}$ as an example, we can write

$$f(\theta) = \sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta).$$

The P_N 's are results of translation from $f_N(\theta)$, hence they are uniformly bounded; by Weierstraß' M-test the series $f(\theta)$ converges uniformly to a continuous periodic function. However, for each P_{N_m} where $m \in \mathbb{N}$, taking S_{2N_m} breaks the symmetry, as

$$|S_{2N_m}(f)(0)| \geqslant c\alpha_m \log N_m,$$

where $c \log N_m$ presents a lower bound as $\left|\tilde{f}_N(0)\right| \geqslant c \log N$ and $S_{2N_m}(f) = \tilde{f}_{N_m}$. As the right-hand side is unbounded, such f indeed has its Fourier series diverging at zero. $\square$

Section 3. The case where f is Lipschitz

In section 2, we proved that there exists a function f by construction where $S_N(f)(0) \to \infty$ as $N \to \infty$. If we wish for asymptotic behavior of $S_N(f)(x)$ at any x, we could simply conduct a translation to diverge $S_N(f)$ at the point of choice. Nevertheless, the Dirichlet kernel isn't fully obsolete when discussing pointwise convergence. In fact, a slightly stronger condition is required for pointwise convergence: *Lipschitz continuity*. We first recall the definition of Lipschitz continuity from 425a.

Definition 7. We say that $f : X \to Y$ is **Lipschitz continuous** there exists a uniform $M \geqslant 0$ such that for every $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leqslant M d_X(x_1, x_2).$$

We first prove a weaker statement, which requires a stronger premise: differentiability.

Theorem 8. Let $f : L^2_{\mathcal{R}}[-\pi, \pi]$ be a function that is differentiable at zero. Then

$$\lim_{N\to\infty} S_N(f)(0) \to f(0).$$

Remark. If f is defined on the circle, $f(-\pi) = f(\pi)$, so differentiability can be defined for every x. Subsequently, the above theorem can be generalized to every point on the circle through translation.

Proof. We first define the function of differentiation at zero. Define $F : (-\pi, \pi) \to \mathbb{R}$ as

$$F(t) = \begin{cases} (f(-t) - f(0))/t, & t \in (-\pi, 0) \cup (0, \pi) \\ -f'(0), & t = 0. \end{cases}$$

Note that $F(t) \to -f'(0)$ as $t \to 0$. As f is differentiable at zero, F is bounded in some neighborhood $(-\delta, \delta)$ of zero. Additionally, for $t \in (-\pi, -\delta] \cup [\delta, \pi)$, $F(t)$ is integrable on the region as f is Riemann integrable and $|t| > \delta$ (therefore $F(t)$ does not explode at any point and discontinuities remain of measure zero.)

Lemma. Let f be bounded on $[a, b]$. If $c \in (a, b)$, and if for all $\delta > 0$ the function is integrable on $[a, c - \delta]$ and $[c + \delta, b]$, then f is integrable on $[a, b]$.

Proof of lemma. Let M denote the bound of f. Choose $\varepsilon > 0$, and let P_1 and P_2 be partitions of $[a, c - \delta]$ and $[c + \delta, b]$, where f is integrable, so that for $i \in [2]$ we have

$$U(P_i, f) - L(P_i, f) < \frac{\varepsilon}{3}.$$

Then take the partition $P = P_1 \cup \{c - \delta\} \cup \{c + \delta\} \cup P_2$, we choose δ small enough so that $\delta 2M < \varepsilon/6$ so that the partitions $\{c - \delta\} \cup \{c + \delta\}$ together integrates to $< \varepsilon/3$. Thus $U(P, f) - L(P, f) < \varepsilon$ as desired.

Because the δ is arbitrary, F is also Riemann integrable by the above lemma, with

$$\int_{-\pi}^{\pi} F(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} \frac{f(-t) - f(0)}{t}\,\mathrm{d}t.$$

Now we return to the statement. We wish to consider the asymptotic behavior of $S_N(f)(0) - f(0)$. Using the unit signed mass property of the Dirichlet kernel, we have, through a common trick in convolutions,

$$
\begin{aligned}
S_N(f)(0) - f(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)D_N(t)\,\mathrm{d}t - f(0) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)D_N(t) - f(0)D_N(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(-t) - f(0)]\,D_N(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)tD_N(t)\,\mathrm{d}t.
\end{aligned}
$$

Here we consider the closed form expression of $D_N(t)$. Particularly,

$$
S_N(f)(0) - f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)t\frac{\sin((N+\frac{1}{2})t)}{\sin(t/2)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F(t)t}{\sin(t/2)}\left[\sin Nt\cos\frac{t}{2} + \cos Nt\sin\frac{t}{2}\right]\mathrm{d}x.
$$

Here we consider an additional lemma covered in 425b lecture.

Lemma. (*Riemann-Lebesgue*) $\hat{f}(n) \to 0$ as $n \to \infty$.

Proof of lemma. The lemma follows directly from Bessel's inequality, which states that the ℓ^2 norm of Fourier coefficients is no greater than the L^2 norm of the function f:

$$
\left\|\hat{f}_n\right\|_{\ell^2} \leqslant \|f\|_{L^2}.
$$

As $f \in L^2_{\mathcal{R}}$, the convergence of the (nonnegative) series imply that $\hat{f}_n \to 0$ (or else the series explodes!)

With Riemann-Lebesgue lemma, we can consider

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F(t)t\cos(t/2)}{\sin(t/2)}\sin Nt\,\mathrm{d}x + \frac{1}{2\pi}\int_{-\pi}^{\pi}(F(t)t)\cos Nt\,\mathrm{d}x.
$$

The functions at the left - $F(t)t\cos(t/2)/\sin(t/2)$ and $F(t)t$ - are both L^2_{R}; therefore the integral tends to zero as $N \to \infty$ as the Fourier coefficients are sums of $\cos Nt$ and $\sin Nt$. This proves the theorem. □

Now consider the above proof, especially where the differentiability is applied:

As f is differentiable at zero, F is bounded in some neighborhood $(-\delta, \delta)$ of zero.

Indeed, if f is Lipschitz at zero instead, F is still bounded in some neighborhood $(-\delta, \delta)$ of zero; although we cannot define $F(t) = -f'(0)$ at $t = 0$ explicitly, considering the equivalence class with a measure-zero differing set brings us to the same conclusion. We state the result here.

Theorem 9. Let $f : L^2_{\mathcal{R}}[-\pi, \pi]$ be a function that is locally Lipschitz at zero. Then

$$
\lim_{N\to\infty} S_N(f)(0) \to f(0).
$$

Subsequently, if f is Lipschitz over the domain, then

$$
S_N(f) \to f;
$$

and on any interval $[a, b]$,

$$
S_N(f) \rightrightarrows f.
$$

Section 4. Cesàro summation and Fejér kernel

In previous sections, we saw that the Dirichlet kernel does not converge at every point, but it has some nice properties (for example, unit signed mass) that we wish to keep when considering an alternative. To do so, we introduce the concept of Cesàro means and the Fejér kernel. Cesàro means allows us to compute an unambiguous limit of series. See the following example.

Example 10. Consider the series

$$s = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots.$$

Its partial sums form the sequence $\{1, 0, 1, 0, \cdots\}$. With two subsequential limits in $\{1, 0\}$, the partial sum has no limit in the classical sense - it diverges.

Definition 11. We define the ***N*-th Cesàro mean** of the sequence $\{s_k\}$, or the ***N*-th Cesàro sum** of the series $s = \sum_{k=0}^{\infty} c_k$, as

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

If $\sigma_N \to \sigma$ as $N \to \infty$, the series $\sum c_k$ is **Cesàro summable** to σ.

Example 12. Observing the above example,

$$\sigma_N = \frac{1 + 0 + \cdots + s_{N-1}}{N} = \begin{cases} \frac{1}{2}, & N \text{ even;} \\ \frac{N+1}{2N}, & N \text{ odd.} \end{cases}$$

The N-thCesàro sum converges to $\frac{1}{2}$, so s is Cesàro summable to $\frac{1}{2}$.

The Cesàro summation criterion is more inclusive than convergence criterion. The example above shows an excample where the series s is Cesàro summable but does not converge; the proposition below gives an explicit proof of the statement.

Proposition 13. If a series converges to s, then it is also Cesàro summable to the same limit s.

Proof. If the series converges to a limit s, its partial sums form a converging sequence. Primarily, for every $\varepsilon > 0$ there exists N^* large such that $N \geqslant N^* \Rightarrow s_N \in (s - \varepsilon, s + \varepsilon)$. Then taking the N-th Cesàro sums, $N \gg N^*$,

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N} = N^{-1} \left(\sum_{k=0}^{N^*} s_k + \sum_{N^*+1}^{N} s_k \right).$$

The second term has $s_k \in (s - \varepsilon, s + \varepsilon)$, hence

$$N^{-1} \sum_{N^*+1}^{N} s_k = \frac{(N - N^*) s_k}{N} \overset{N \to \infty}{\to} (s - \varepsilon, s + \varepsilon),$$

whereas the first term has constant N^*-th partial sum, so $N^{-1}(\cdot)$ vanishes as N tends to infinity. As the choice of ε is arbitrary, $\sigma_N \to s$ as $N \to \infty$, as desired. □

We then consider the N-th Cesàro mean of the Fourier series. By the definition of the Dirichlet kernel, we have

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} = \frac{(D_0 * f)(x) + \cdots + (D_{N-1} * f)(x)}{N}.$$

Because convolutions are linear, we can write

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where $F_N(x)$ is the N-th *Fejér kernel*: an average of the Dirichlet kernels.

Definition 14. The N-th **Fejér kernel** is defined as

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

Proposition 15. The N-th Fejér kernel exhibits a closed form in

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

Proof. Considering the closed form of the n-th Dirichlet kernel, $n \in [N]$,

$$D_n(x) = \frac{\sin((n+\frac{1}{2})x)}{\sin(x/2)}.$$

(Note that here we used lowercased n because we need the capitalized N for the Fejér kernel.) Then

$$NF_N(x) = \sum_{n=0}^{N-1} D_n = \frac{\sum_{n=0}^{N-1}\sin((n+\frac{1}{2})x)}{\sin(x/2)} = \frac{\sum_{n=0}^{N-1}\sin((n+\frac{1}{2})x)\sin(x/2)}{\sin^2(x/2)}.$$

We utilize the product-to-sum formula. For each $n \in [N-1]_0$,

$$\sin((n+\frac{1}{2})x)\sin\frac{x}{2} = \frac{1}{2}\left(\cos nx - \cos(n+1)x\right).$$

Then the numerator eliminates to $\frac{1}{2}(\cos 0 - \cos Nx) = \frac{1}{2}(1-\cos Nx)$. Applying double angle formula,

$$NF_N(x) = \frac{\sin^2(Nx/2)}{\sin^2(x/2)},$$

as desired. □

Section 5. Pointwise convergence of Fejér kernel

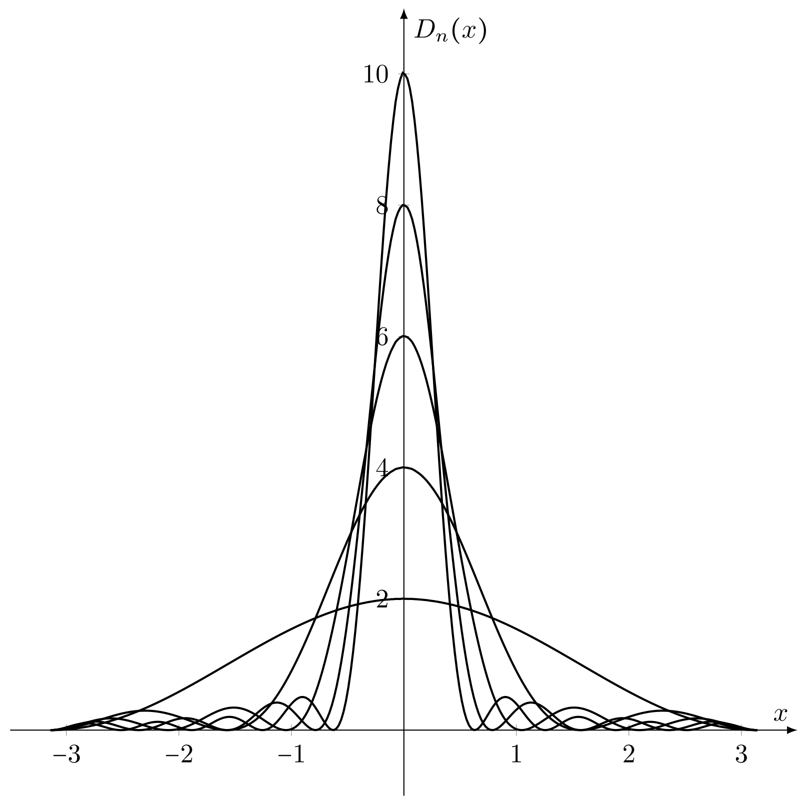
The interesting and important property of the Fejér kernel is that it *is* an approximation identity. This property directly results in Cesàro summability of integrable functions, which, in turn, allows for a number of important results in Fourier series.

Proposition 16. The Fejér kernel $\{F_N\}_N$ is an approximate identity.

Proof. Recall the definition of the Fejér kernel:

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

We can observe the behavior of the Fejér kernel, just like the case for Dirichlet kernel.



We see that the Fejér kernel, including the squared sine terms, no longer exhibits oscillations in negative y. Therefore we only need to check for (signed) unit mass and asymptotic culminating behavior at origin. For the signed unit mass, we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,\mathrm{d}x = \sum_{n=0}^{N-1}\frac{1}{2N\pi}\underbrace{\int_{-\pi}^{\pi} D_n(x)\,\mathrm{d}x}_{=2\pi} = 1.$$

Regarding the culminating behavior, we wish to prove that

$$\lim_{N\to\infty}\left[\int_{-\pi}^{\delta} + \int_{\delta}^{\pi}\right] F_N(x)\,\mathrm{d}x = 0.$$

Indeed, consider the closed form of the Fejér kernel, we observe that for

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)},$$

$\sin^2(x/2)$, a continuous function, exhibits a positive infimum on the compact set $[-\pi,-\delta]\cup[\delta,\pi]$ for every $\delta > 0$; denote it c_δ. On the other hand, $\sin^2(Nx/2)$ is bounded by ± 1; therefore we have that

$$\int_{\delta\leqslant|x|\leqslant\pi} |F_N(x)|\,\mathrm{d}x \leqslant \int_{\delta\leqslant|x|\leqslant\pi}\frac{1}{Nc_\delta}\,\mathrm{d}x \overset{N\to\infty}{\to} 0.$$

Therefore $\{F_N\}_N$ is indeed an approximate identity. □

We can now establish the result of pointwise convergence in terms of Fejér kernel and Cesàro summability.

Theorem 17. If $f \in L^2_{\mathcal{R}}[-\pi,\pi]$, then the Fourier series of f is Cesàro summable to f at every point of continuity of f. Additionally, if f is 2π-periodic and continuous, then the Fourier series of f is uniformly Cesàro summable to f.

Proof. As the Fejér kernel is an approximate identity, its convolution with any bounded, locally Riemann integrable f approaches f at every point of continuity of f. We can therefore write, for $x \in [-\pi,\pi]$ where f is continuous,

$$\lim_{N\to\infty} (F_N * f)(x) \to f(x).$$

As $(F_N * f)(x) = \sigma_N(f)(x)$, we see that the N-th Cesàro sum of f indeed converges to $f(x)$. The claim for uniform summability similarly is derived from the results of approximate identity: if f is continuous on an open (a,b) then for any compact $[c,d] \subset (a,b)$ the uniform convergence holds:

$$\lim_{N\to\infty} F_N * f \rightrightarrows f.$$

Now as f is continuous and periodic, taking any continuous interval of length $> 2\pi$ and take a 2π-length subset suffices for uniform convergence. □

Lastly, we restate a theorem that has been proven in 425b lecture. However, now equipped with knowledge of Cesàro summability, we can now view the theorem in a different angle.

Corollary 18. The class of continuous periodic functions $C_{\text{per}}[-\pi,\pi]$ can be uniformly approximated by trigonometric integrals; the trigonometric polynomials are dense in $C_{\text{per}}[-\pi,\pi]$.

Proof. Note that the Cesàro means are trigonometric polynomials themselves. As the Fourier series of f is uniformly summable to f, take the trigonometric polynomials defined by the Cesàro means suffices. □

Section 6. The Gibbs Phenomenon

In this section, we talk a bit about the *Gibbs's phenomenon*, which states that near a jump discontinuity, the Fourier series of a function overshoots (or undershoots) it by approximately 9% of the jump. We consider the following problem regarding the sawtooth function.

Problem: Stein Ex. 3.20. Let $f(x)$ denote the sawtooth function (as defined in proposition 6). The Fourier series of f is

$$\hat{f}(x) = \frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n} = \sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

and f has a jump discontinuity at the origin with $f(0^+) = \pi/2$, $f(0^-) = -\pi/2$. Show that

$$\max_{0<x\leqslant\pi/N} S_N(f)(x) - \frac{\pi}{2} = \int_0^{\pi}\frac{\sin t}{t}\,\mathrm{d}t - \frac{\pi}{2} \approx 0.09\pi.$$

Proof. We apply the integral expansion of $\hat{f}(x)$. Specifically, we claim that

$$\sum_{n=1}^{N}\frac{\sin nx}{n} = \frac{1}{2}\int_0^x (D_N(t)-1)\;\mathrm{d}t.$$

Considering the integrand $D_N(t)-1$, we have that

$$D_N(t) - 1 = \sum_{n=-N}^{N} e^{int} - e^0 = \sum_{n=-N}^{-1} e^{int} + \sum_{n=1}^{N} e^{int} = \sum_{n=1}^{N} e^{int} + e^{-int}.$$

Again, we utilize the fact that $z+\overline{z} = 2\mathrm{Re}\, z$ for complex $z\in\mathbb{C}$ to conclude

$$D_N(t) - 1 = \sum_{n=1}^{N} 2\cos nt.$$

Considering the linearity of integrals, we further attempt to align the equation for each n; consider the right-hand side, the proof follows from a simple high school integration.

$$\frac{1}{2}\int_0^x 2\cos nt\;\mathrm{d}t = n^{-1}\left[\sin nt\right]_0^x = \frac{\sin nx}{n},$$

as desired.

To take the maximum of the expression, we consider the first derivative of the integral; define

$$\varphi_N(x) = S_N(f)(x) - \frac{\pi}{2},$$

we then have, by the fundamental theorem of calculus,

$$\varphi_N'(x) = S_N'(f)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\int_0^x (D_N(t)-1)\;\mathrm{d}t\right) = \frac{D_N(x)-1}{2}.$$

Note that for $\varphi_N' = x$, we need $D_N(x) = 1$ for $x\in(0,\pi/N]$. Now taking the closed form expression of the Dirichlet kernel, we then need to solve

$$\sin((N+\frac{1}{2})x) = \sin\frac{x}{2}.$$

Now as we don't expect the two arguments to be equal, we think about the solution where

$$(N+\frac{1}{2})x + \frac{x}{2} = \pi \Rightarrow x = \frac{\pi}{N+1},$$

and we observe that x lies within the desired range of $x \in (0, \pi/N]$. Evaluating the maximum value,

$$\begin{aligned}
\varphi_N(\pi/(N+1)) &= \frac{1}{2}\int_0^{\frac{\pi}{N+1}} \left(\frac{\sin(N+\frac{1}{2})t)}{\sin(t/2)} - 1\right) \mathrm{d}t - \frac{\pi}{2} && \text{(Closed form of Dirichlet kernel)} \\
&= \frac{1}{2}\int_0^{\pi\frac{N+1/2}{N+1}} \left(\frac{\sin t}{\sin(t/(2N+1))} - 1\right)\left(\frac{1}{N+1/2}\right) \mathrm{d}t - \frac{\pi}{2} && \text{(Transformation } t \mapsto (N+1/2)t\text{)} \\
&\overset{N\to\infty}{\to} \frac{1}{2}\int_0^{\pi} \frac{(2N+1)\sin t}{(N+1/2)t} \mathrm{d}t - \frac{\pi}{2} = \int_0^{\pi} \frac{\sin t}{t} \mathrm{d}t - \frac{\pi}{2} \approx 0.09\pi, && \text{(Taking limit as } N \to \infty\text{)}
\end{aligned}$$

as desired. □

The implication here is that $S_N(f)$ reaches a maximum value of approximately 0.59π near zero, and a minimum value of approximately -0.59π on the other side. Although increasing N improves the L^2 difference between f and $\hat{f}$ on $[-\pi, \pi]$, the overshooting (and undershooting) behavior of Fourier series persists.

The figure below shows the Gibbs's phenomenon, although for another function - the "square wave", defined as -1 for $x \in (-\pi, 0)$, 1 for $x \in (0, \pi)$. The "square" nature of the square wave provides a more obvious outlook of the Gibbs's phenomenon. We see that although the Fourier series for $N = 30$ approximates the square wave well in terms of "overall fit" - an indicator of mean-square convergence, near the jumps, at 0, for example, the series still overshoots - by approximately 0.09π.

Gibbs Phenomenon (N=30)