Chapter 1 - 1st order differential equations

Differential equation: any equation that involves one or more derivatives of an unknown function

Linear DE: (order n)

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = F(x) \quad \text{(all functions of } x \text{ only)}$$

Malthusian population model: $\frac{dP}{dt} = kP \Rightarrow P(t) = Ce^{kt}$

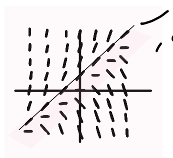
Logistic population model: $\frac{dP}{dt} = k(1-\frac{P}{C})P$

Newton's law of cooling: $\frac{dT}{dt} = -k(T-T_m) \Rightarrow T(t) = T_m + Ce^{-kt}$

Existence and uniqueness theorem:

Let $\frac{dy}{dx} = f(x,y)$ be a function that is continuous on the rectangle $R = \{(x,y): x \in [a,b], y \in [c,d]\}$.
Suppose that $\frac{\partial f}{\partial y}$ is continuous on R. Then for any point (x_0, y_0) in the rectangle R, there exists an interval I containing x_0 s.t. the initial-value problem has a unique solution in I.

$g(x,y) = y - x$.



isocline: family of curves with same slope $\frac{dy}{dx} = y-x = 1$.

Slope field: sketch of $\frac{dy}{dx} = y - x \Rightarrow$ general shape of solution curves

Equilibrium solution: any solution in the form $y(x) = y_0$.
- No other solution curves intersect the eq. solution.

Separable DE: $p(y)\frac{dy}{dx} = q(x) \Rightarrow \int p(y)\,dy = \int q(x)\,dx$

Linear DE: $\frac{dy}{dx} + p(x)y = q(x) \Rightarrow (I(x)y)' = I(x)q(x)$
- $I(x) = e^{\int p(x)dx}$

Chapter 2 - Matrices and Systems of Linear Equations

Linear equations: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

Matrix: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ m rows, n columns; a_{mn}: element (of a matrix) $\longleftrightarrow$ $A^T = \begin{bmatrix} a^T_{11} & a^T_{12} & \cdots & a^T_{1m} \\ a^T_{21} & a^T_{22} & \cdots & a^T_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^T_{n1} & a^T_{n2} & \cdots & a^T_{nm} \end{bmatrix}$ n rows, m columns

$A = B \Leftrightarrow a_{ij} = b_{ij} \quad \forall\, 1 \le i \le m,\ 1 \le j \le n$

$a_{ij} = a^T_{ji} \quad \forall\, 1 \le i \le m,\ 1 \le j \le n$

Square matrix: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ (main diagonal)

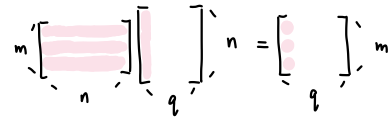
- $Tr(A) = a_{11} + a_{22} + \dots + a_{nn}$
- A is upper triangular $\Leftrightarrow$ elements are 0 when below main diagonal
- A is lower triangular $\Leftrightarrow$ elements are 0 when above main diagonal
- A is diagonal $\Leftrightarrow$ elements are 0 when not on diagonal

Symmetric matrix: $A^T = A$.

Skew-symmetric matrix: $A^T = -A$.

Matrix function: $\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{bmatrix}$

Multiplication of matrices:



$C = AB \Rightarrow c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, $1 \le i \le m, 1 \le k \le p$

Matrix multiplication is not commutative.

$n \times n$ identity matrix: $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

- $I_nA = AI_n = A$, when defined.

Transpose properties:
(1) $(A^T)^T = A$
(2) $(A+C)^T = A^T + C^T$
(3) $(AB)^T = B^TA^T$

System of linear equations of m equations and n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

if $b_i = 0\ \forall\, 1 \le i \le m$, system is homogeneous
else, system is nonhomogeneous

- No solution: system is inconsistent
- Has solution: system is consistent

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $\quad A^{\#} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

Row-echelon matrices follow three properties:
1. If there are any rows consisting entirely of zeros, they are grouped together at the bottom
2. The first nonzero element in any nonzero row is 1
3. The leading 1 of any row is on the right of the leading 1 of the row above.

Elementary row operations:
1. P_{ij} - permute i^{th} and j^{th} rows of A.
2. $M_i(k)$ - multiply the i^{th} row of A by a factor of k.
3. $A_{ij}(k)$ - add $k \cdot i^{th}$ row of A to j^{th} row of A.

After finitely many EROs, the new matrix is row-equivalent to A.

Pivot position:

Rank: number of nonzero rows in row-echelon form

Reduced row-echelon form:
1. It is a row-echelon matrix
2. Any column that contains a leading 1 has 0 everywhere else.

Solution of linear system $Ax = b$: $\begin{cases} \text{if } r < r^{\#}, \text{ system is inconsistent} \\ \text{if } r = r^{\#}, \text{ system is consistent, and:} \end{cases}$
- Unique solution iff $r^{\#} = n$.
- Infinitely many solution iff $r^{\#} < n$.

Inverse of a matrix A^{-1}: if there exists a matrix A^{-1} s.t. $AA^{-1} = A^{-1}A = I_n$, then A is invertible, and A^{-1} is the inverse of A.

Invertible matrices are nonsingular, noninvertible matrices are singular.

If A^{-1} exists, then $Ax = b$ has the solution $x = A^{-1}b$ for every b.

If A and B are invertible, then:
- A^{-1} is invertible and $(A^{-1})^{-1} = A$.
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Elementary matrix is a matrix obtained from performing one ERO to I_n.

Premultiplying A by an elementary matrix is equivalent to performing the respective ERO.

$E_kE_{k-1}\cdots E_2E_1A = I_n \Rightarrow A^{-1} = E_kE_{k-1}\cdots E_2E_1$.
$\Rightarrow A = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$.

The multiplier is the multiple of a specific row that is subtracted from row i to get a zero in the (i,j) position.

LU factorization:

$A = LU \begin{cases} U = E_kE_{k-1}\cdots E_2E_1A \\ L = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}. \end{cases}$

Solving $Ax = b$: $A = LU \Rightarrow Ly = b$
$\Rightarrow Ux = y$

Chapter 3 - Determinants

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow Det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Area of parallelogram $\begin{cases} \vec{a} = \langle a_1, a_2 \rangle \\ \vec{b} = \langle b_1, b_2 \rangle \end{cases}$: $A = \left\| \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix} \right\|$

Volume of parallelepiped $\begin{cases} \vec{a} = \langle a_1, a_2, a_3 \rangle \\ \vec{b} = \langle b_1, b_2, b_3 \rangle \\ \vec{c} = \langle c_1, c_2, c_3 \rangle \end{cases}$: $V = \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\|$

Determinant of triangular matrix: $\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11}a_{22}\cdots a_{nn}$.

ERO effects on determinant:
- Permutation: $\det(B) = -\det(A)$
- Multiplication: $\det(B) = k\det(A)$
- Addition: $\det(B) = \det(A)$

A is invertible $\Leftrightarrow \det(A) \neq 0$.

$Ax = 0$ has infinitely many solutions iff $\det(A) = 0$,
one trivial solution iff $\det(A) \neq 0$.

$\det(kA) = k^n\det(A)$.
$\det(A^T) = \det(A)$.
$\det(A) = \det(B) + \det(C)$.
$\det(A) = 0$ if any column or row is zero.
$\det(A) = 0$ if any column or row is a multiple of another
$\det(AB) = \det(A)\det(B)$.
$\det(A^{-1}) = \frac{1}{\det(A)}$

cofactor: $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is obtained from removing i^{th} row and j^{th} column

cofactor expansion theorem: multiply the elements in any row/column by their cofactors, and their sum is $\det(A)$.

Adjoint: replacing every element by its cofactor, then taking transpose. M_c^T. $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$

Cramer's rule: $x_k = \frac{\det(B_k)}{\det A}$