MATH 225 Homework 6

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Problem 1: Goode 7.1.3

Verify that $\lambda = 3$ and $v = (2, 1, -1)$ are an eigenvalue/eigenvector pair for the matrix $A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$.

Solution. We use the equation $Av = \lambda v$. Here,

$$Av = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Problem 2: Goode 7.1.7

Given that $v_1 = (1, -2)$ and $v_2 = (1, 1)$ are eigenvectors of $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, determine the eigenvalues of A.

Solution. Consider $Av = \lambda v$. For $v_1 = (1, -2)$, $Av = (2, -4) \Rightarrow \lambda_1 = 2$. For $v_2 = (1, 1)$, $Av = (5, 5) \Rightarrow \lambda_2 = 5$.

Problem 3: Goode 7.1.18

Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

Solution. The characteristic polynomial is $\lambda^2 - 2\lambda + 5$, setting it to zero gives $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$. Now consider $A_1 = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}$ and $A_2 = \begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix}$. Solving the coefficient matrix respectively gives the eigenvectors $v_1 = r(1 + i, 2)$ and $v_2 = s(1 - i, 2)$.

Problem 4: Goode 7.1.21

Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Solution. The characteristic polynomial is $(3-\lambda)(\lambda^2 - 4\lambda + 3)$, giving eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$.

Now consider $A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$. Solving the coefficient matrix respectively gives the eigenvectors $v_1 = r(0,1,1)$ and $v_2 = s(0,1,-1)$.

Problem 5: Goode 7.1.34

Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

(a) Show that the characteristic polynomial of A is $p(\lambda) = \lambda^2 - 5\lambda + 6$.

(b) Show that A satisfies its characteristic equation. That is, $A^2 - 5A + 6I_2 = 0_2$. (This result is known as the Cayley-Hamilton Theorem and is true for general $n \times n$ matrices.)

(c) Use the result from (b) to find A^{-1}.

Solution. We solve the problem step by step.

(a) $A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$. The characteristic equation is then $\det(A - \lambda I) = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6$.

(b) Consider $A^2 - 5A + 6I_2 = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1-5+6 & -5+5+0 \\ 10-10+0 & 14-20+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) $(A^{-1})(A^2 - 5A + 6I_2) = A - 5I_2 + 6A^{-1} = 0$. This gives

$$A^{-1} = \frac{1}{6}(5I_2 - A) = \frac{1}{6}\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/6 \\ -1/3 & 1/6 \end{bmatrix}.$$

Problem 6: Goode 7.1.35

(a) Determine all eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

(b) Reduce A to row-echelon form, and determine the eigenvalues of the resulting matrix. Are these the same as the eigenvalues of A?

Solution. We solve the problem step by step.

(a) The characteristic equation is $\lambda^2 + \lambda - 6 = 0$, giving $\lambda_1 = 2$, $\lambda_2 = -3$.

(b) $A \sim \begin{bmatrix} 1 & 2 \\ 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The characteristic equation is then $(1-\lambda)^2 = 0$, giving $\lambda = 1$. It is clearly not the same as the eigenvalues of A.

Problem 7: Goode 7.1.43

Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues.

Proof. It suffices to show that $\det(A^T - \lambda I) = \det(A - \lambda I)$.
From $\det(A) = \det(A^T)$ for all A, we have

$$\begin{aligned} \det(A^T - \lambda I) &= \det(A^T - \lambda I)^T \\ &= \det((A^T)^T - \lambda I^T) \\ &= \det(A - \lambda I). \end{aligned}$$

□

Problem 8: Goode 7.2.4

Determine the multiplicity of each eigenvalue and a basis for each eigenspace of $A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}$. Hence, determine the dimension of each eigenspace and state whether the matrix is defective or nondefective.

Solution. A has a characteristic equation of $\lambda^2 - 6\lambda + 9 = 0$, giving $\lambda = 3$ of multiplicity 2. The eigenvector is $v = r(1,1)$, so the basis for the eigenspace is $(1,1)$. The dimension is 1, hence A is defective.

Problem 9: Goode 7.2.7

Determine the multiplicity of each eigenvalue and a basis for each eigenspace of $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{bmatrix}$. Hence, determine the dimension of each eigenspace and state whether the matrix is defective or nondefective.

Solution. A has a characteristic equation of $(4-\lambda)(\lambda^2 - 3\lambda - 4) = 0$, giving $\lambda = 4$ of multiplicity 2 and $\lambda = -1$ of multiplicity 1. Here we check the eigenvalue $\lambda = 4$. $A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{bmatrix}$. We then see that $\text{rank}(A - 4I) = 1$, implying that $\dim(E) = 2$ for $\lambda = 4$. Geometric multiplicity matches algebraic multiplicity, so A is nondefective.

Problem 10: Goode 7.2.13

Determine the multiplicity of each eigenvalue and a basis for each eigenspace of $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$. Hence, determine the dimension of each eigenspace and state whether the matrix is defective or nondefective.

Solution. A has a characteristic equation of $-\lambda^3 + 2\lambda^2 = 0$, giving $\lambda = 0$ of multiplicity 2 and $\lambda = 2$ of multiplicity 1. Here we check the eigenvalue $\lambda = 0$. $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We then see that rank($A$) = 2, implying that $\dim(E) = 1$ for $\lambda = 0$. Geometric multiplicity does not match algebraic multiplicity, so A is defective.

Problem 11: Goode 7.2.19

Determine whether $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$ is defective or nondefective.

Solution. Consider the characteristic equation $\lambda^2 - 4\lambda + 13 = 0$. As $\Delta < 0$, we have two distinct imaginary eigenvalues, hence two eigenvectors corresponding to each eigenvalue. We can then conclude that A is nondefective.

Problem 12: Goode 7.2.21

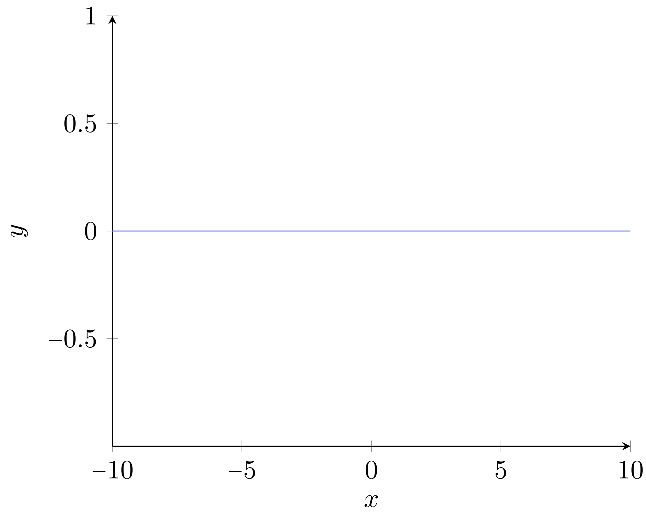
Determine whether $A = \begin{bmatrix} -1 & 2 & 2 \\ -4 & 5 & 2 \\ -4 & 2 & 5 \end{bmatrix}$ is defective or nondefective.

Solution. We first know that A has only one eigenvalue $\lambda = 3$ of multiplicity 3. We start to suspect if A is defective or not. We then observe that $A - 3\lambda = \begin{bmatrix} -4 & 2 & 2 \\ -4 & 2 & 2 \\ -4 & 2 & 2 \end{bmatrix}$ whose rank is 1, so the eigenspace is a plane of dimension 2. The geometric multiplicity does not match algebraic multiplicity, we can then conclude that A is defective.

Problem 13: Goode 7.2.25

Determine a basis for each eigenspace of $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ and sketch the eigenspaces.

Solution. We first observe that $\lambda = 2$ with multiplicity 2. Consequently, the eigenvector is $(1, 0)$, so the two-dimensional space has an eigenspace spanned by the $(1, 0)$ vector. On a two-dimensional cartesian spane, the eigenspace is the line $y = 0$, as follows:



Problem 14: Goode 7.2.33

Use the result of problem 32 to determine the sum and the product of the eigenvalues of $A = \begin{bmatrix} -1 & -2 & 0 \\ 6 & -3 & -8 \\ -2 & 2 & 1 \end{bmatrix}$.

Solution. We know from problem 32 that $\det(A)$ is the product of the eigenvalues of A, and $\operatorname{tr}(A)$ is the sum of the eigenvalues of A. We have $\operatorname{tr}(A) = -3$, so the eigenvalues sum to -3. We also have $\det(A) = -1(13) - (-2)(-10) = -33$, so the eigenvalues have product of -33.

Problem 15: Goode 7.3.3

Determine whether $A = \begin{bmatrix} -7 & 4 \\ -4 & 1 \end{bmatrix}$ is diagonalizable. Where possible, find a matrix S such that $S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$.

Solution. A has an eigenvalue $\lambda = -3$ of multiplicity 2. It then is associated with one eigenvector, and the eigenspace is then of dimension 1, so A is defective. It follows that A is not diagonalizable.

Problem 16: Goode 7.3.4

Determine whether $A = \begin{bmatrix} 1 & -8 \\ 2 & -7 \end{bmatrix}$ is diagonalizable. Where possible, find a matrix S such that $S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$.

Solution. A has two eigenvalue $\lambda = -3$ of multiplicity 2. It then is associated with one eigenvector, and the eigenspace is then of dimension 1, so A is defective. It follows that A is not diagonalizable.

Problem 17: Goode 7.3.10

Determine whether $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ is diagonalizable. Where possible, find a matrix S such that $S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Solution. A has an eigenvalue $\lambda = -2$ of multiplicity 1, and an eigenvalue $\lambda = -1$ of multiplicity 2. From the coefficient matrix, the first eigenspace is formed by the intersection of two linearly independent planes, so it is a line. Similarly, the second eigenspace is also formed by the intersection of two linearly independent planes, so it is also a line. Geometric multiplicity does not match algebraic multiplicity, so A is not diagonalizable.

Problem 18: Goode 7.3.19

Use the ideas introduced in this section to solve the following system of differential equations:

$$x_1' = 6x_1 - 2x_2, \quad x_2' = -2x_1 + 6x_2.$$

Solution. Consider $x' = Ax$, where $A = \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$. The transformed system is $y' = (S^{-1}AS)y$, where $x = Sy$. To determine S, we need the eigenvalues and eigenvectors. The characteristic polynomial is $p(\lambda) = \lambda^2 - 12\lambda + 32$. From this we know $v = r(1, -1)$ for $\lambda = 4$, $v = (1, 1)$ for $\lambda = 8$. We then have a S where $S^{-1}AS = \operatorname{diag}(4, 8)$, so $y_1' = 4y_1$ and $y_2' = 8y_2$. This gives $x = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{4t} \\ c_2 e^{8t} \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} - c_2 e^{8t} \\ c_1 e^{4t} + c_2 e^{8t} \end{bmatrix}$.