MATH 425b Lecture Notes

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Spring 2023

Beginning of January 10, 2023

In an usual course of analysis I, we are expected to learn:

- $\mathbb{R}$ and $\mathbb{C}$;
- Topology of metric spaces, including open/closed sets, continuity, compactness, etc.
- Sequences, series, and convergence tests of series
- Single-variable calculus
- *Uniform convergence*, which is particularly important.

And below is a list of what we are going to do this semester. Depending on the progress with analysis I, the first two sections (power series and improper integrals) may or may not be a review.

- Power series and special functions
- Improper Riemann integral and convex functions
- Function spaces and *approximation theory* (long-story-short definition: approximating some function that looks *bad* with some functions that are better for us to treat)
- Fourier series (if time permits)
- Multivariable calculus: derivative (as a linear transformation), multiple integration, differential forms, and perhaps the most important concept of multivariable calculus, Stokes' theorem.

The emphasis of analysis I is on **the nuts and bolts of problem solving**. When we think of a mathematical problem, we have a starting point and an ending point, and our goal is to find the logical path that connects the two. Meanwhile, the problems in analysis II usually involves a higher degree of logical sophistication. This certainly involves more practice and more sophisticated strategization.

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Definition. A power series of the variable z in $\mathbb{R}$ or $\mathbb{C}$ centered at a with coefficients c_n for $n \in \mathbb{N}_0$, takes the form of

$$\sum_{n=0}^{\infty} c_n (z-a)^n.$$

The above power series has radius of convergence R of α^{-1} for $0 < \alpha < \infty$, where

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|}.$$

Of course, if $\alpha = +\infty$ then $R = 0$; if $\alpha = 0$ then $R = +\infty$.

Proposition. Given $(c_n)_{n=0}^{\infty}$ in $\mathbb{C}$, and suppose the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ has radius of convergence $R > 0$. Then the series converges absolutely when $|z-a| < R$ and diverges when $|z-a| > R$. Additionally, the series converges uniformly on $\{z : |z-a| \leqslant r\}$ for any $r \in (0, R)$.

Proof. For the first two parts, we utilize the root test. Consider

$$\sqrt[n]{|c_n (z-a)^n|} = |z-a| \sqrt[n]{|c_n|},$$

where its $\limsup$ has value $R^{-1}|z-a|$. With the root test, we see that the limit superior is less than unity if $|z-a| < R$, and greater than unity if $|z-a| > R$. This proves the conditional (absolute) convergence and divergence. For the last part regarding uniform convergence, we use the Weierstraß' M-test.

Theorem. The Weierstraß' M-test: $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly if $|f_n(z)| \leqslant M_n$ and $\sum_{n=0}^{\infty} M_n < +\infty$.

Pick a $r \in (0, R)$. With $|c_n (z-a)^n| \leqslant c_n r^n$, we use the root test to obtain

$$\limsup_{n \to \infty} \sqrt[n]{|c_n r^n|} = \left(\limsup_{n \to \infty} \sqrt[n]{|c_n|} \right) r = \frac{r}{R} < 1,$$

hence the series converges by the Weierstraß' M-test. □

Remark. Applying the uniform limit theorem to the sequence of partial sums $\left(\sum_{n=0}^{N} c_n (z-a)^n \right)_{N=1}^{\infty}$, we obtain the continuity of $\sum_{n=0}^{\infty} c_n (z-a)^n$ inside the disc of convergence. Specifically, we can choose an arbitrary point in the disc of convergence, then there exists a ball where its closure includes the point, and by the last part of the previous proposition we can then apply the uniform limit theorem to show the continuity of the series at that point.

We now introduce the concept of a Cauchy product. Consider two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, and suppose we want to find their "product" $\sum_{n=0}^{\infty} c_n z^n$. If we simply multiply the power series of $f(z)g(z)$, we get

$$c_n = \sum_{k+l=n} a_k b_l = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k.$$

Definition. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their Cauchy product is

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

Two questions can be raised regarding the Cauchy product.

- Does the Cauchy product converge? More specifically, what are the conditions for it to converge? (Merten's)
- If $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$ and $\sum_{n=0}^{\infty} c_n = C$. Does $AB = C$ hold? In other words, if the Cauchy product converges, does it have to converge to the product AB? (Able's)

Example. The first question might seem obvious. Consider

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}.$$

The Cauchy product is then $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k a_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}$.
In fact, $c_n \nrightarrow 0$, hence $\sum_{n=0}^{\infty} c_n$ must not converge. More to cover next class.

Beginning of January 11, 2023

In the last lecture, we ended up with a counterexample for the convergence of Cauchy product given convergence of two series. We will continue with that example today.

Example. Consider $a_n = \frac{(-1)^n}{\sqrt{n+1}}$. The Cauchy product of $\sum_{n=0}^{\infty} a_n$ with itself has terms

$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k-1}}.$$

Claim. $c_n \nrightarrow 0$ as $n \to \infty$, so the Cauchy product *cannot* converge.

Proof of claim. Consider the inequality $2ab \leqslant a^2 + b^2$. With $a = \sqrt{k+1}$ and $b = \sqrt{n-k+1}$, we can conclude that $a^2 + b^2 = n + 2$, and we can construct a nonzero lower bound of $|c_n|$ with

$$|c_n| \geqslant \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2} \nrightarrow 0.$$

Merten's theorem connects convergence of series and convergence of Cauchy product.

Theorem. (*Merten*) Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely to A, and $\sum_{n=0}^{\infty} b_n$ converges to B. Then their Cauchy product converges to $C = AB$.

Proof. Consider the partial sums, denoted as $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$. We have

$$C_n = c_0 + c_1 + \ldots + c_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + \ldots + a_n b_0).$$

Regrouping the terms of C_n gives $C_n = a_0(b_0 + \ldots + b_n) + a_1(b_0 + \ldots + b_{n-1}) + \ldots + a_n b_0$. Setting $\beta_n = B - B_n = \sum_{k=n+1}^{\infty} b_k$,

$$C_n = a_0(B - \beta_n) + a_1(B - \beta_{n-1}) + \ldots + a_n(B - \beta_0) = (a_0 + \ldots + a_n)B - (a_0\beta_n + a_1\beta_{n-1} + \ldots + a_n\beta_0).$$

Notice that $a_0 + \ldots + a_n = A_n$, denote the tail as γ_n and computation gives $C_n = A_n B - \gamma_n$. Given $A_n \to A$, it suffices to show $\gamma_n \to 0$ (then $C_n \to AB - 0 = AB$). Note that this *could* happen as the terms of γ_n becomes smaller and smaller. We split the expression into two parts to utilize the infinisimality of a_n and β_n:

$$|\gamma_n| = \sum_{k=0}^{N}|a_{n-k}\beta_k| \leqslant \sum_{k=0}^{N}|a_{n-k}\beta_k| + \sum_{k=N+1}^{n}|a_{n-k}||\beta_k| \leqslant \max\{|a_{n-N}|,\dots,|a_n|\}\sum_{k=0}^{N}|\beta_k| + \max\{|\beta_{N+1}|,\dots,|\beta_n|\}\sum_{k=0}^{n-N-1}|a_k|.$$

When $n \to \infty$, the tail behaviors of β_n and a_n and the boundedness of the finite sums guarantee convergence. □

Theorem. (*Able*) Assume $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$. Assume also that their Cauchy product $\sum_{n=0}^{\infty} c_n = C$. Then $AB = C$.

To prove the theorem, we need to assume the validity of the lemma. (Which isn't really easy to prove...)

Lemma. Suppose $\sum_{n=0}^{\infty} a_n$ converges. Define $f : (-1, 1] \to \mathbb{C}$, and $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $f(1-) = \sum_{n=0}^{\infty} a_n$.

Proof. Define $f(x)\sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$, $h(x) = \sum_{n=0}^{\infty} c_n x^n$. As $h(x)$ is the Cauchy product of $f(x)$ and $g(x)$, for $x \in (-1, 1)$, both series converge *absolutely*, hence Merten's theorem implies $h(x) = f(x)g(x)$. Taking the limit of the expression as $x \to 1^-$ gives $C = h(1-) = f(1-)g(1-) = AB$. □

Proof of lemma. Consider each a_n as the difference of partial sums: $a_n = S_n - S_{n-1}$ (with $s_{-1} = 0$.) Then

$$f(x) = \sum_{n=0}^{\infty}(s_n - s_{n-1})x^n = \sum_{n=0}^{\infty} s_n x^n - \sum_{n=1}^{\infty} s_{n-1}x^n = \sum_{n=0}^{\infty} s_n x^n - \sum_{n=0}^{\infty} s_n x^{n+1} = \sum_{n=0}^{\infty} s_n x^n(1-x).$$

Note that we actually *cheated* a bit; we could *not* guarantee the convergence of the two series (but it could be proven with some steps.) With s_n convergent, it is bounded. More about it in the next lecture.

Beginning of January 13, 2023

Last class we ended with a (rather incomplete) proof of Abel's theorem. However, we are missing an auxiliary lemma. Here we complete the work left yesterday.

Lemma. Assuming $\sum_{n=0}^{\infty} a_n$, define $f : (-1, 1] \to \mathbb{C}$ by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $f(1-) = f(1)$.

Proof. Consider each a_n as the difference of partial sums: $a_n = S_n - S_{n-1}$ (with $s_{-1} = 0$.) Then

$$f(x) = \sum_{n=0}^{\infty}(s_n - s_{n-1})x^n = \sum_{n=0}^{\infty} s_n x^n - \sum_{n=1}^{\infty} s_{n-1}x^n = \sum_{n=0}^{\infty} s_n x^n - \sum_{n=0}^{\infty} s_n x^{n+1} = \sum_{n=0}^{\infty} s_n x^n(1-x).$$

Note that we actually *cheated* a bit; we could *not* guarantee the convergence of the two series (but it could be proven with some steps.) As we want to investigate the tail behavior $|f(x) - s|$, take

$$\sum_{n=0}^{\infty} a_n = s = s\frac{1-x}{1-x} = s(1-x)\sum_{n=0}^{\infty} x^n.$$

Thus,

$$|f(x) - s| = \left|(1-x)\left(\sum_{n=0}^{\infty} s_n x^n - s\sum_{n=0}^{\infty} x^n\right)\right|.$$

Estimating the expression above, we use the method of "splitting the sums" again to bound the expression above as follows:

$$|f(x) - s| \leqslant (1-x)\left|\sum_{n=0}^{N}(|s| + M) + \sup_{n\geqslant N+1}|s - s_n|\sum_{n=0}^{\infty}|x|^n\right| \leqslant (1-x)(N+1)(N+M) + \sup_{n\geqslant N+1}|s - s_n|$$

In part 1, we utilize the fact that the left-hand sum (with terms) $s_n x^n$ is bounded by M, and the right-hand sum has terms no bigger than 1, hence the expression itself is no larger than $|s|$. In part 2, we begin with the sum starting with $n = N + 1$, but it is surely no greater than the sum starting with $n = 0$. Beginning with $n = 0$ allows us to eliminate terms. Now we can first set N large enough such that $\sup_{n\geqslant N+1}|s-s_n| < \frac{\epsilon}{2}$ (and we can do it due to the convergent tail behavior of s_n.) Then we can choose x close to 1 (using the δ-ϵ method) such that $(1-x)(N+1)(N+M) < \frac{\epsilon}{2}$. Then the upper bound is established and we can conclude the result.

Remark. The best takeaway of the proof above would be the trick of "splitting the sums".

Now we discuss about the double sums. Generally, we are interested in the above question: when is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{m,n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_{m,n} = \sum_{l=0}^{\infty}\sum_{m+n=l} x_{m,n}?$$

The above three situations are obtained when we take the double sum (which can be represented as a matrix) via three different ways: horizontally (grouping rows first), vertically (grouping columns first), and diagonally (grouping diagonals). It should be rather easily believed that the statement doesn't hold all the time.

Theorem. "If we have absolute convergence, we can reorder things." More specifically, suppose $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|x_{m,n}| < +\infty$. Then $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{m,n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_{m,n} = \sum_{l=0}^{\infty}\sum_{m+n=l}^{\infty} x_{m,n}$.

Proof. Define $y_m = \sum_{n=0}^{\infty}|x_{m,n}|$ (grouping columns) and $z_n = \sum_{m=0}^{\infty}|x_{m,n}|$ (grouping rows). By assumption y_m 's and z_n 's are bounded. Consider additionally a mapping $s_m : \mathbb{N}_0 \cup \{+\infty\} \to \mathbb{C}$ defined as $s_m(N) = \sum_{n=0}^{N} x_{m,n}$. Thus $(s_m)_{m=0}^{\infty}$ is a sequence $(s_m)_{m=0}^{\infty}$ of continuous functions. Note that $s_m(N) \leqslant y_m$ for all $m \in \mathbb{N}$. Then by Weierstraß' M-test, $\sum_{m=0}^{\infty} s_m(N) \to t$, which is continuous on positive infinity as well. The continuity gives

$$t(+\infty) = \lim_{N\to\infty} t(N) = \lim_{N\to\infty}\sum_{m=0}^{\infty} s_m(N) = \lim_{N\to\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{N} x_{m,n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{m,n}.$$

On the other hand, the definition of t implies directly that $t(+\infty) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_{m,n}$.

More to cover about the diagonal sums in next lecture. □

Beginning of January 18, 2023

We begin today by recalling a theorem from last class regarding exchanging double sums.

Theorem. If $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_{m,n}| < \infty$, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{m,n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_{m,n} = \sum_{l=0}^{\infty}\sum_{j+k=l} x_{j,k}.$$

Having proven the first equality in the last lecture, we will attempt to prove the second equality now. Again, we consider the notation that $(y_m)_{m=0}^{\infty} = \sum_{n=0}^{\infty}|x_{m,n}|$, $(z_n)_{n=0}^{\infty} = \sum_{m=0}^{\infty} a_n|x_{m,n}|$.

Proof. Note that for every $\epsilon > 0$, there exists $N \in \mathbb{N}_0$ such that $\sum_{m>N} y_m < \frac{\epsilon}{2}$ and $\sum_{n>N} z_n < \frac{\epsilon}{2}$. Thus for such N,

$$|A - \sum_{m=0}^{N}\sum_{n=0}^{N} x_{m,n}| \leqslant \sum_{m>N} y_m + \sum_{n>N} z_n < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(Here A is the desired common limit.) Geometrically, we desire the whole $N \times N$ box to be contained in some partial sums of diagonals. That is, we want to find a L large such that $\sum_{l=0}^{L}\sum_{j+k=l} x_{j,k}$ includes the $N \times N$ box of our choice. Note that by choosing such N we already made the terms outside the box "negligibly small". As

$L = 2N$ would suffice for our needs, we can have the following expression, which is bounded above by ϵ:

$$|\sum_{l=0}^{2N}\sum_{j+k=l} x_{j,k} - \sum_{m=0}^{N}\sum_{n=0}^{N} x_{m,n}| \sum_{m>N} y_m + \sum_{n>N} z_n < \epsilon.$$

Here, only the $N \times N$ square contains the terms that *do* matter. The difference between the two double sums for large N and large L then tends to zero. This proves the claim. $\square$

Now we discuss further on power series and their differentiability, starting with the following proposition.

Proposition. Let $(c_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$. Suppose $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $R > 0$. Define $(-R, R) \to \mathbb{C}$ by $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Then,

- $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly to f on $[-r, r]$ for every $r \in (0, R)$.
- f is differentiable on $(-R, R)$, with $f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$.

Additionally, the radius of convergence of $f'(x)$ is also R. (Similar statement holds for $f^{(n)}(x)$.)

Proof. Consider the radius of convergence of $\sum_{n=1}^{\infty} nc_n x^{n-1}$ first, which can be obtained with the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|nc_n|} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1 \cdot R^{-1} = R^{-1}.$$

Regarding the differentiability, first define $f_N(x) = \sum_{n=0}^{N} c_n x^n$. As the partial sum is finite, we have

$$f_N'(x) = \sum_{n=1}^{N} nc_n x^{n-1}.$$

As $f_N \rightrightarrows f$ on $[-r, r]$, we know that $f_N' \rightrightarrows g$, where $g(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$ on the same interval by the differentiable limit theorem (covered in 425a). $\square$

A direct consequence of the above proposition: functions that can be represented by power series are very special. In particular, $f(x) = \sum_{n=0}^{\infty} c_n x^n$ are infinitely many times differentiable on the interval of convergence. Additionally,

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots \Rightarrow c_0 = f(0),\ c_1 = f'(0),\ c_2 = \frac{f''(0)}{2},\ c_3 = \frac{f'''(0)}{6}, \ldots$$

Generalizing the c_n 's, $c_n = f^{(n)}(0)/n!$. This is the Maclaurin series of $f(x)$ (at $x = 0$.) More generally, the Taylor series of $f(x)$ at $x = a$ takes the form of

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Remark. The power series representation is unique. Specifically, in the radius of convergence, if

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

then $a_n = b_n$ for all $n \in \mathbb{N}$. This can be shown by the fact that $\sum_{n=0}^{\infty}(a_n - b_n)x^n = 0$, hence $a_n - b_n = 0^{(n)} = 0$.

Remark. Consider the function $f(x) = e^{-1/x^2}$ at $x \neq 0$ and 0 if $x = 0$. We will prove it in a homework exercise that $f^{(n)}(0) = 0$ for all n, but f does not have a power series representation centered at zero.

We end today's class by touching on Taylor's theorem.

Definition. Let U and V be open subsets of $\mathbb{R}$ or $\mathbb{C}$, with $V \subset U$. Consider a function $f : U \to \mathbb{C}$. f is analytic on V if at every $a \in V$, f has a representation $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ on some neighborhood centered at a.

Theorem. (*Taylor*) Suppose $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $R > 0$. Define $f : B_{\mathbb{C}}(0,R) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then for every $a \in B_{\mathbb{C}}(0,R)$, there exists $(d_n)_{n=0}^{\infty}$ such that

$$z \in B_{\mathbb{C}}(a, R - |a|) \Rightarrow f(z) = \sum_{n=0}^{\infty} d_n (z-a)^n.$$

Beginning of January 20, 2023

Today we will first discuss improper integrals. We have already learned Riemann integrals, which are usually defined on a compact domain $[a,b]$. We can generalize the notation to non-compact intervals, which brings up the definition of locally integrable functions.

Definition.
if I is an interval in $\mathbb{R}$, we let $\mathcal{R}_{\text{loc}}(I)$ denote the space of locally integrable functions on I. That is, the set of those functions f defined in I which are Riemann integrable on every compact subinterval $[a,b] \subset I$.

Definition. If $f \in \mathcal{R}_{\text{loc}}([a,\infty))$. The improper integral of the first kind is defined by

$$\int_a^{\infty} f(x)\, \mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x)\, \mathrm{d}x.$$

If $g \in \mathcal{R}_{\text{loc}}((c,b])$, the improper integral of the second kind is defined by

$$\int_c^b g(x)\, \mathrm{d}x = \lim_{a \to c^+} \int_a^b g(x)\, \mathrm{d}x.$$

Definition. If the improper integral of $|f|$ converges, then the integral of f converges absolutely. If f converges but not absolutely, it converges conditionally.

Remark. Any improper integral of the first kind can be transformed into an improper integral of the second kind, and vice versa. For example,

$$\int_a^{\infty} f(x)\, \mathrm{d}x, \;\; x \mapsto \frac{1}{y} \Rightarrow \int_0^{\frac{1}{a}} \frac{1}{y^2} f\left(y^{-1}\right)\, \mathrm{d}y.$$

Definition. If "something goes wrong" at the point $c \in [a,b]$ (perhaps it's unbounded or not included in the domain,) and we wish to *not* consider the point c, we can define the improper integral $\int_a^b g(x)\, \mathrm{d}x$ as

$$\int_a^b g(x)\, \mathrm{d}x = \int_a^c g(x)\, \mathrm{d}x + \int_c^b g(x)\, \mathrm{d}x.$$

Remark. If $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ and $\int_{-\infty}^{\infty} f(y)\, \mathrm{d}y$ converges, then

$$\int_{-\infty}^{\infty} f(y)\, \mathrm{d}y = \lim_{R \to \infty} \int_{x-R}^{x+R} f(y)\, \mathrm{d}y.$$

The following lemma is analogous to the comparison test for series of numbers.

Lemma. Assume $f \in \mathcal{R}_{\text{loc}}([a, \infty))$ is nonnegative. Then

- Either $\int_a^\infty f(x)\,\mathrm{d}x$ converges, or $\int_a^x f(y)\,\mathrm{d}y \to \infty$ as $x \to +\infty$. In particular, the limit exists in $\overline{\mathbb{R}}$.

- (Integral comparison test) If $\int_a^\infty f(x)\,\mathrm{d}x$ converges and $g \in \mathcal{R}_{\text{loc}}([a, \infty))$ satisfies $|g(x)| \leqslant f(x)$ for all $x \in [a, +\infty)$, then $\int_a^\infty g(x)\,\mathrm{d}x$ converges as well.

Proof of (2). Consider the case where g is nonnegative. Then the limit exists in $\overline{\mathbb{R}}$ by (1). Then convergence of $\int_a^\infty f(x)\,\mathrm{d}x$ implies $F(x) = \int_a^x f(y)\,\mathrm{d}y$ is bounded above. Then for all $x \geqslant a$,

$$0 \leqslant \int_a^x g(y)\,\mathrm{d}y \leqslant F(x) \leqslant M \leqslant +\infty.$$

Thus the limit $\lim_{x\to\infty} \int_a^x g(y)\,\mathrm{d}y$ is finite, hence the improper integral converges. Moreover, if g is real-valued but not necessarily nonnegative, take the positive and negative parts of g and apply the same method. □

Corollary. Assume $h \in \mathcal{R}_{\text{loc}}([a, \infty))$, and assume $\int_a^\infty h(x)\,\mathrm{d}x$ converges absolutely. Then $\int_a^\infty h(x)\,\mathrm{d}x$ converges.

Proof. Apply the previous lemma with $f = |h|$ and $g = h$. □

Proposition. Let f be a positive, monotonically decreasing function on $[0, +\infty)$. Then for $m, M \in \mathbb{N}$, $m < M$, we have

$$\sum_{n=m+1}^{M+1} f(n) \leqslant \int_m^M f(x)\,\mathrm{d}x \leqslant \sum_{n=m}^{M} f(n).$$

Consequently, the integral $\int_0^\infty f(x)\,\mathrm{d}x$ converges if and only if $\sum_{n=0}^\infty f(n)$ does.

Proof. $f(n+1) \leqslant \int_n^{n+1} f(x)\,\mathrm{d}x \leqslant f(n)$ as f is monotonically decreasing. Summing over n from m to M proves the statement directly. □

Note that the previous proposition is only useful for proving absolute convergence. To show conditional convergence, consider the following theorem.

Theorem. Let $f : [a, \infty) \to \mathbb{R}$ be continuous and let $g : [a, \infty) \to \mathbb{R}$ be continuously differentiable. Assume additionally that $F(x) := \int_a^x f(y)\,\mathrm{d}y$ is bounded on $[a, \infty)$, $g(x) \to 0$ as $x \to \infty$, and $\int_a^\infty |g'(y)|\,\mathrm{d}y$ converges. Then $\int_a^\infty f(x)g(x)\,\mathrm{d}x$ converges.

Proof. We utilize the integration by parts formula.

$$\int_a^b f(x)g(x)\,\mathrm{d}x = \int_a^b F'(x)g(x)\,\mathrm{d}x = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)\,\mathrm{d}x.$$

Since $\lim_{b\to\infty} g(b) = 0$ and F is bounded, the first term $F(b)g(b)$ tends to zero. Furthermore, $F(a) = 0$ so the second term is zero. To show that $\int_a^\infty F(x)g'(x)\,\mathrm{d}x$ converges, let $|F|$ be bounded by B. Thus $|F(x)g'(x)| \leqslant B|g'(x)|$. Since $\int_a^\infty |g'(x)|\,\mathrm{d}x$ converges, $\int_a^\infty F(x)g'(x)\,\mathrm{d}x$ converges as well. □

Corollary. Assume $g : [a, +\infty) \to \mathbb{R}$ nonnegative, $g(x)$ monotonically decreasing to zero. Assume also that g is continuously differentiable. Then $\int_a^\infty g(x)\sin x \, \mathrm{d}x$ and $\int_a^\infty g(x)\cos x \, \mathrm{d}x$ both converge.

Proof. Apply the previous theorem with $f(x) = \sin x$ or $g(x) = \cos x$. Note that as g is monotonically decreasing,

$$\int_a^b |g'(x)| \, \mathrm{d}x = -\int_a^b g'(x) \, \mathrm{d}x = g(a) - g(b) \to g(a).$$

□

Beginning of January 23, 2023

Today we will continue our discussion of improper integrals before finishing our proof on Taylor's theorem, discussed last Wednesday. Last Friday we noted that series and improper integrals have many similarities. However, there also exist differences. Namely, the convergence of $\int_a^\infty f(x) \, \mathrm{d}x$ does *not* imply $f(x) \to 0$.

Example. $\int_1^\infty \cos(x^2) \, \mathrm{d}x$ converges, but $\lim_{x\to\infty} \cos(x^2) \nrightarrow 0$.

We will throw a new concept in: the indicator function.

Definition. The indicator function of a set A is defined by

$$\mathbb{I}_A(x) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$$

Example. Consider the case where

$$f(x) = \sum_{k=1}^\infty \mathbb{I}_{[k-\frac{1}{2k^2}, k+\frac{1}{2k^2}]}(x)\sqrt{k}.$$

Then $f(k) \to \infty$ for $k \in \mathbb{N}$, $k \to \infty$ (the function is actually unbounded), yet

$$\int_0^\infty |f(x)| \, \mathrm{d}x = \int_0^\infty f(x) \, \mathrm{d}x = \sum_{k=1}^\infty \int_{k-\frac{1}{2k^2}}^{k+\frac{1}{2k^2}} \sqrt{k} \, \mathrm{d}x = \sum_{k=1}^\infty k^{-3/2} < +\infty.$$

The series converges absolutely but f is unbounded!

Now we go back to discuss Taylor's theorem. Recall that f is analytic in V if it admits a power series representation with radius of convergence $R > 0$ centered at any point $a \in V$. With this definition, we have Taylor's theorem as follows:

Theorem. (*Taylor*) Suppose $\sum_{n=0}^\infty c_n z^n$ has radius of convergence $R > 0$. Define $f : B_{\mathbb{C}}(0, R) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^\infty c_n z^n$. Then for every $a \in B_{\mathbb{C}}(0, R)$, there exists a sequence d_n such that $f(z) = d_n(z-a)^n$ whenever $|z-a| < R - |a|$.

Proof. We start our proof by noting that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n((z-a)+a)^n = \sum_{n=0}^{\infty} c_n \left(\sum_{k=0}^{n} \binom{n}{k} (z-a)^k a^{n-k} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_n \binom{n}{k} (z-a)^k a^{n-k}.$$

Here we wish to change the order of the double sums. Assuming validity of change in order,

$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} c_n \binom{n}{k} a^{n-k} (z-a)^k.$$

Here treating the blue part as d_k gives us a new power series approximation.
Now it suffices to show that $\sum_{n=0}^{\infty} \sum_{k=0}^{n} |c_n \binom{n}{k} (z-a)^k a^{n-k}|$ is finite. Regarding this, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{n} |c_n \binom{n}{k} (z-a)^k a^{n-k}| &\leqslant \sum_{n=0}^{\infty} \sum_{k=0}^{n} |c_n| \binom{n}{k} |z-a|^k |a|^{n-k} \\ &= \sum_{n=0}^{\infty} |c_n| (|z-a| + |a|)^n. \end{aligned}$$

With $|z-a| + |a| < R$, the infinite series converges (absolutely), so we can justify the change of order of sums. □

For the sake of completeness, we touch on some special functions that are defined based on the power series.

Definition. The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is defined by the power series as

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Based on the definition of exponential, we then have the definition of the sine function and the cosine function:

Definition. The sine function $\sin : \mathbb{C} \to \mathbb{C}$ and the cosine function $\cos : \mathbb{C} \to \mathbb{C}$ are defined as

$$\sin z = \frac{1}{2i} \left[\exp(iz) - \exp(-iz) \right], \quad \cos(z) = \frac{1}{2} \left[\exp(iz) + \exp(-iz) \right].$$

Example. $\exp \overline{z} = \overline{\exp z}$. This is equivalent to $\sum_{n=0}^{\infty} \frac{(\overline{z})^n}{n!} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Using the (uniform) continuity of the complex mapping $z \mapsto \overline{z}$, defining $f(z) = \overline{z}$ and applying limits give

$$\sum_{n=0}^{\infty} \frac{\overline{z}^n}{n!} = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{z^n}{n!} = \lim_{N \to \infty} f\left(\sum_{n=0}^{N} \frac{z^n}{n!} \right) = f\left(\lim_{N \to \infty} \frac{z^n}{n!} \right) = \overline{\exp z}.$$

Example. $\sin z = \mathrm{Im}(\exp(iz))$ and $\cos z = \mathrm{Re}(\exp(iz))$ if $z \in \mathbb{R}$.

Remark. From the above example we have

$$e^{i\theta} = \mathrm{cis}(\theta) = \cos\theta + i \sin\theta.$$

We end the class with a calculation.

Proposition.

$$\exp(z+w) = \exp(z)\exp(w).$$

Proof.

$$\begin{aligned}\exp(z+w) &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \exp(z)\exp(w).\end{aligned}$$

□

Beginning of January 25, 2023

Today we will talk about convex functions.

Definition. Given a real vector space V (here we consider $\mathbb{R}^n$), a subset $E \subset V$ is convex if whenever $a, b \in E$ and $\lambda \in [0,1]$, we have $\lambda a + (1-\lambda)b \in E$ as well. $\lambda a + (1-\lambda b)$ is the convex combination of a and b. A function $f : E \to \mathbb{R}$ is convex if its epigraph $\operatorname{epi}(f) = \{(\mathbf{x}, y) : y \geqslant f(\mathbf{x})\}$ is a convex subset of $V \times \mathbb{R}$. (Here $\mathbf{x}$ can be a vector - it need not be in $\mathbb{R}$.)

Proposition. $f : E \to \mathbb{R}$ is convex if and only if for every $a, b \in E$ and for every $\lambda \in [0,1]$, $f(\lambda a + (1-\lambda b) \leqslant \lambda f(a) + (1-\lambda) f(b)$.

We will restrict attention to functions defined on an intevral $(\alpha, \beta) \subset \overline{\mathbb{R}}$.

Lemma. Let $f : (\alpha, \beta) \to \mathbb{R}$ be a convex function, $\alpha < a < b < c < \beta$. Then

$$\frac{f(b)-f(a)}{b-a} \leqslant \frac{f(c)-f(a)}{c-a} \leqslant \frac{f(c)-f(b)}{c-b}.$$

Proof. Let $b = \lambda a + (1-\lambda)c$ for some $\lambda \in (0,1)$. Thus

$$f(b) = f(\lambda a + (1-\lambda)c) \leqslant \lambda f(a) + (1-\lambda) f(c).$$

Subtracting $f(a)$ from the expression gives

$$f(b) - f(a) \leqslant (\lambda - 1) f(a) + (1-\lambda) f(c) = (1-\lambda)(f(c) - f(a)).$$

At last,

$$\frac{f(b)-f(a)}{b-a} \leqslant \frac{(1-\lambda)(f(c)-f(a))}{(1-\lambda)(c-a)}.$$

Here we used the fact that $b = \lambda a + (1-\lambda)c = (1-\lambda)(c-a)$. □

Lemma. Let $f : (\alpha, \beta) \to \mathbb{R}$ be a function. If

$$\frac{f(b)-f(a)}{b-a} \leqslant \frac{f(c)-f(b)}{c-b}$$

whenever $\alpha < a < b < c < \beta$, then f is convex.

Remark. This is essentially the converse of the previous lemma.

Proof. We want to show that for every $a, c \in (\alpha, \beta)$, (WLOG let $a < c$,) $f(\lambda a + (1-\lambda)c) \leqslant \lambda f(a) + (1-\lambda)f(c)$. Choosing $b = \lambda a + (1-\lambda)c$, there exists $\gamma \in \mathbb{R}$ such that

$$\frac{f(b) - f(a)}{b - a} \leqslant \gamma \leqslant \frac{f(c) - f(b)}{c - b}.$$

From the first inequality, we have $f(b) \leqslant f(a) + \gamma(b - a)$; from the second inequality, $f(b) \leqslant f(c) - \gamma(c - b)$. Thus

$$f(b) \leqslant \lambda f(a) + (1-\lambda)f(c) + \gamma[\lambda(b-a) - (1-\lambda)(c-b)].$$

As the blue part is $\lambda(b-a) - (1-\lambda)(c-b) = (\lambda + (1-\lambda))b - (\lambda a + (1-\lambda)c) = 0$, essentially

$$f(b) \leqslant \lambda f(a) + (1-\lambda)f(c),$$

which suffices to show that f is convex. □

Introducing a new concept: Lipschitz functions. The concept of Lipschitz may or may not be completely new; (it in fact *did* appear on Prof. Leslie's Midterm 2 for 425a.) But anyways... Here it is, the more formal definition.

Definition. A function $f : E \to \mathbb{R}$ is Lipschitz if there exists $L > 0$ such that for every $x, y \in E$,

$$|f(x) - f(y)| \leqslant L(|x - y|.$$

$f : E \to \mathbb{R}$ is locally Lipschitz of $f|_K$ is Lipschitz for any compact $K \subset E$.

Remark. $f \in C^1$ if f is continuously differentiable.

Remark. A couple of remarks in order:

- $f \in C^1(E) \Rightarrow f \in \mathrm{Lip}_{\mathrm{loc}}(E) \not\Rightarrow f \in \mathrm{Lip}(E)$.
- $f \in \mathrm{Lip}(E) \Rightarrow f$ is uniformly continuous.
- $f \in \mathrm{Lip}(E) \not\Rightarrow f$ is differentiable. (However, it is differentiable *almost everywhere*.)

We end today with a sketch of a proposition.

Proposition. If $f : (\alpha, \beta) \to \mathbb{R}$ is convex, then $f \in \mathrm{Lip}_{\mathrm{loc}}((\alpha, \beta))$.

We give a brief sketch of the proof: choose $K \subset (\alpha, \beta)$ compact. Then our goal is to show that $|\frac{f(y)-f(x)}{y-x}| \leqslant L$. More to cover in the next lecture.

Beginning of January 27, 2023

In the last lecture, we discussed convex functions - functions with a convex epigraph (the region *above* the graph). We also discussed that convex combinations satisfy

$$f(\lambda a + (1-\lambda)b) \leqslant \lambda f(a) + (1-\lambda)f(b),$$

where the equality holds when f is linear. Additionally, the secant line between a and b has a lower slope than the secant line between c and d (for $a < b < c < d$ in domain.)

Also recall that Lipschitz functions obey the following inequality:

$$|f(x) - f(y)| \leqslant L(x, y)$$

for some $L > 0$ for all x, y in the domain. The function is locally Lipschitz if f is Lipschitz on any compact set.

Corollary. If $f : (\alpha, \beta) \to \mathbb{R}$ is convex, it is locally Lipschitz.

Proof. Given K compact and $K \subset (\alpha, \beta)$, it suffices to show that for all $x, y \in K$, there exists L_K such that

$$|f(x) - f(y)| \leqslant L_K |x - y|.$$

Choose $a, b, c, d \in (\alpha, \beta)$ such that $a < b < \inf K < \sup K < c < d$. Then, for all $x, y \in K$ and $x < y$ (WLOG),

$$\frac{f(b) - f(a)}{b - a} \leqslant \frac{f(y) - f(x)}{y - x} \leqslant \frac{f(d) - f(c)}{d - c}.$$

Thus

$$L_K = \max\left\{\left|\frac{f(b) - f(a)}{b - a}\right|, \left|\frac{f(d) - f(c)}{d - c}\right|\right\} \Rightarrow \left|\frac{f(y) - f(x)}{y - x}\right| \leqslant L_K.$$

□

Proposition. If $f : (\alpha, \beta) \to \mathbb{R}$ is differentiable, then f is convex if and only if f' is nondecreasing.

Proof. ($\Rightarrow$) Assume f is convex. Then WLOG choose $x < y$; from the convexity of f we have, for some $h > 0$,

$$\frac{f(x) - f(x - h)}{h} \leqslant \frac{f(y + h) - f(y)}{h}.$$

As differentiability of f is assumed, taking $h \to 0$ gives $f'(x) \leqslant f'(y)$.

($\Leftarrow$) Assume f' is nondecreasing. Choose p and q arbitrarily. By the mean value theorem, there exists $a, b, c \in (\alpha, \beta)$ such that

$$f'(p) = \frac{f(b) - f(a)}{b - a}, \quad f'(q) = \frac{f(c) - f(b)}{c - b}, \quad f'(p) \leqslant f'(q).$$

As p and q are arbitrary, the proof is done. □

We now try to apply convexity in three inequalities: Young, Hölder, and Minkowski.

Theorem. (*Young*) Let $a, b > 0$, $\lambda \in (0, 1)$, $p = \lambda^{-1}$ and $q = (1 - \lambda)^{-1}$ (p and q are Hölder conjugates.) Then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof.

$$\begin{aligned}
ab &= \exp(\log ab) \\
&= \exp(\log a + \log b) \\
&= \exp\left(p^{-1} \log a^p + q^{-1} \log b^q\right) \\
&\leqslant \lambda \exp(\log a^p) + (1 - \lambda) \exp(\log b^q) \\
&= \frac{a^p}{p} + \frac{b^q}{q}.
\end{aligned}$$

Convexity is used in the inequality step; note that $p^{-1} = \lambda$ and $q^{-1} = 1 - \lambda$. $\square$

Theorem. (*Hölder*) Let $f, g \in \mathcal{R}_{\text{loc}}(\mathbb{R})$; p and q are Hölder conjugates. Then, (proof as exercise,)

$$\int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x \leqslant \left(\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} = \|f\|_p \|g\|_q .$$

We will later learn that $(\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x)^{\frac{1}{p}}$ is the L^p norm of f. The following theorem shows that this is indeed a norm. In other words, the triangle inequality is satisfied.

Theorem. (*Minkowski*) Let $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ and $p > 1$. Then

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p .$$

In more humane (as of now) notations,

$$\left(\int_{\mathbb{R}} |f(x) + g(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \leqslant \left(\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |g(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}$$

Proof. Consider $\|f + g\|_p^p$ first. First consider the triangular inequality $|f + g| \leqslant |f| + |g|$. This way,

$$\begin{aligned} I = \int_{\mathbb{R}} |f(x) + g(x)|^p\,\mathrm{d}x &\leqslant \int_{-\infty}^{\infty} (|f(x)| + |g(x)|)\,|f(x) + g(x)|^{p-1}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} |f(x)|\,|f(x) + g(x)|^{p-1}\,\mathrm{d}x + \int_{\mathbb{R}} |g(x)|\,|f(x) + g(x)|^{p-1}\,\mathrm{d}x. \end{aligned}$$

By Hölder's,

$$I \leqslant \left(\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \underline{\left(\int_{\mathbb{R}} |f(x) + g(x)|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}} + \left(\int_{\mathbb{R}} |g(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \underline{\left(\int_{\mathbb{R}} |f(x) + g(x)|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}}$$

Thus

$$I = \int_{\mathbb{R}} |f(x) + g(x)|^p\,\mathrm{d}x \leqslant \left[\left(\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |g(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\right] \left(\int_{\mathbb{R}} |f(x) + g(x)|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}} .$$

Simplification gives the desired result. $\square$

Lastly, we talk a bit about $\log$-convexity.

Definition. $f : (\alpha, \beta) \to (0, \infty)$ is $\log$-convex if $\log \circ f$ is convex.

A couple remarks are in order.

Remark. It is (to be proven as an exercise) true that $\log$-convex implies convex.

Remark. Perhaps the *least* $\log$-convex function is the exponential function; $\log(\exp(x)) = x$ (which is the *least* convex function.)

Proposition. If $f(\alpha, \beta) \to (0, \infty)$ is $\log$-convex, then, (proof as exercise,)

$$f(\lambda a + (1 - \lambda)b) \leqslant f(a)^{\lambda} f(b)^{1-\lambda}.$$

Beginning of January 30, 2023

Today we will discuss about the Gamma function. The Gamma function provides an extension of $n \to n!$ to $(0, \infty)$. If we are to extend the factorial function in a meaningful way, we must consider the following two properties:

- $(\Gamma 1)$ $\Gamma(x+1) = x\Gamma(x)$,
- $(\Gamma 2)$ $\Gamma(1) = 1$.

It turns out that there are many extensions satisfying $(\Gamma 1)$ and $(\Gamma 2)$. Adding one additional constraint makes the function unique.

- $(\Gamma 3)$ Γ is $\log$-convex.

It makes natural sense; considering the $\log$ of the factorial function,

$$\frac{\log((n+1)!) - \log(n!)}{(n+1) - n} = \log\left(\frac{(n+1)!}{n!}\right) = \log(n+1),$$

an increasing function.

Theorem. There exists at most one $f : (0, \infty) \to \mathbb{R}$ satisfying $(\Gamma 1) - (\Gamma 3)$.

Proof. Let's assume first that some f satisfies $(\Gamma 1) - (\Gamma 3)$. To include the $\log$-convexity, it is better to consider $\varphi = \log f$; it then suffices to show the convexity of φ. Considering $f(n+1) = n!$, $\varphi(n+1) = \log n!$, and the convexity of f guarantees

$$\log n \leqslant \frac{\varphi(n+1+x) - \varphi(n+1)}{x} \leqslant \log(n+1) \Rightarrow 0 \leqslant \varphi(n+1+x) - \varphi(n+1) - x\log n \leqslant x\log\left(\frac{n+1}{n}\right).$$

Noting $\varphi(n+1) = \log(n!)$, the inequality becomes

$$0 \leqslant \varphi(n+1+x) - \log\left(n!n^x\right) \leqslant x\log\left(1+n^{-1}\right).$$

Consider $\varphi(n+1+x)$, which has the form

$$\varphi(n+1+x) = \log(f(n+1+x)) = \log((n+x)f(n+x)) = \ldots = \log((n+x)(n-1+x)\ldots(1+x)xf(x)) = \varphi(x)+\log(x(1+x)\ldots(n+x)).$$

With such expansion in mind, we can rewrite the equation as

$$0 \leqslant \varphi(x) - \log\left(\frac{n!n^x}{x+(1+x)\ldots(n+x)}\right) \leqslant x\log\left(1+n^{-1}\right).$$

Subsequently,

$$\varphi(x) = \lim_{n\to\infty} \log\left(\frac{n!n^x}{x(1+x)\ldots(n+x)}\right).$$

Here $f(x)$ would now be defined on $(0,1)$; however, $(\Gamma 1)$ and $(\Gamma 2)$ guarantees that as long as f is defined on $(0,1]$ it would be defined on $\mathbb{R}_+$. This finishes the proof. $\square$

Definition. The gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

Proposition. Γ satisfies $(\Gamma 1)$ – $(\Gamma 3)$.

Proof. First consider

$$x\Gamma(x) = \int_0^\infty x t^{x-1} e^{-t}\,\mathrm{d}t = \int_0^\infty t^x e^{-t}\,\mathrm{d}t = \Gamma(x+1).$$

This satisfies $(\Gamma 1)$. For $(\Gamma 2)$, $\Gamma(1) = \int_0^\infty t^0 e^{-t}\,\mathrm{d}t = 1$. $(\Gamma 3)$ is left as homework. □

We will end the section with Stirling's formula. The formula itself doesn't need great attention, but it's useful to know nevertheless.

Theorem. (*Stirling*)

$$\lim_{x\to+\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1.$$

Corollary. For large x,

$$\Gamma(x+1) \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi x}.$$

That wraps up chapter 2 (of the lecture notes.) In the next chapter, we will deal with vector spaces and some of their properties. The following content may (or may not) be review. Here we will predefine a field F ($F = \mathbb{R}$ or $F = \mathbb{C}$ for the scope of this course.)

Definition. A vector space over F, a F-vector space, is a set V with two operations:

- Addition $V \times V \to V$,
- Multiplication by scalars $F \times V \to V$.

Operations in the F-vector space must satisfy commutative and associative properties of addition, associative and distributive properties of multiplication (in both ways - scalar multiple and vector), and the existence of an additive identity (the "zero") and a multiplicative identity (the "one").

Definition. A subspace W of an F-vector space is a subset $W \subset V$ which is itself a F-vector space.

Remark. We only need to check that W is closed under addition and scalar multiplication.

Example. $F^n = F \times F \times \ldots \times F$ is an F-vector space.

Definition. Consider a F-vector space V, and $S \subset V$. A linear combination of elements of S is a sum

$$\sum_{i=1}^n c_i v_i, \quad c_i \in F, v_i \in S, i \neq j \Rightarrow v_i \neq v_j.$$

A linear combination is trivial if all the c_i 's are zero; otherwise it is nontrivial.
S is linearly dependent if there exists a nontrivial linear combination such that $\sum_{i=1}^n c_i v_i = \mathbf{0}$. Otherwise, S is linearly independent. The span of S, denoted $\mathrm{span}(S)$, is the set of all linear combinations of S.

Definition. S is a (Hamel) basis for V if for every $v \in V$ there exists a unique finite linear combination of elements of S which is equal to v.

A couple remarks in order, recalling from concepts of linear algebra, namely the invertible matrix theorem.

Remark. S is a Hamel basis if and only if S spans V and S is linearly independent.

Remark. If S is a basis for V and S is finite, V is finite-dimensional with $\dim V = \#S$. Otherwise, V is infinite-dimensional.

Remark. Suppose V is an n-dimensional F-vector space, $n \in \mathbb{N}$. Then

- any n-tuple of linear independent elements of V is a basis for V,
- any n-tuple in V that spans V is a basis for V,
- any basis for V consists of n elements of V.

Definition. If $B = (v_1, \ldots, v_n)$ is a basis for an F-vector space V, then for $L_B : V \to F^n$,

$$L_B\left[c_1 v_1 + \ldots + c_n v_n\right] = (c_1, \ldots, c_n)$$

is the coordinate map associated to B.

Beginning of February 1, 2023

A reminder of last lecture: given a field $(F, +_F, \cdot_F)$, a vector space over F, or an F-vector space, is a pair $(V, +_V, \cdot_V)$ such that V is closed under addition $+_V : V \times V \to V$ and scalar multiplication $\cdot_V : F \times V \to V$ and is compatible with the field operations in F.

In order to do analysis on vector spaces, we need a topology; and the most common way to introduce a topology is by introducing a norm, measuring the *size* of an element in a vector space.

Definition. Let V be an F-vector space. A seminorm on V is a function $\|\cdot\| : V \to [0, \infty)$ such that

- $\|v\| \geqslant 0$ for all $v \in V$,
- $\|\alpha v\| = |\alpha| \, \|v\|$ for all $\alpha \in F$ and $v \in V$,
- (Triangle inequality) $\|u + v\| \leqslant \|u\| + \|v\|$ for all $u, v \in V$.

Additionally, a norm on V is a seninorm such that

- $\|v\| = 0$ if and only if $v = 0$.

Example. Consider $F^n = F \times \dots \times F$, where $F = \mathbb{R}$ or $\mathbb{C}$. The Euclidean norm is defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{|x_1|^2 + \dots + \|x_n\|^2}.$$

Meanwhile, the "square" norm, or the uniform norm, is defined by

$$\|(x_1, \dots, x_n)\|_u = \max_{1 \leqslant j \leqslant n} |x_j|.$$

Example. Consider F^X, the set of functions $f : X \to F$. Let

$$B(X; F) = \left\{ f \in F^X, f \text{ bounded} \right\}.$$

The uniform norm $\|\cdot\|_u : B(X; F) \to [0, \infty)$ is defined by

$$\|f\|_u = \sup_{x \in X} \|f(x)\|.$$

When $X = [n]$, the uniform norm is exactly the one above.

Remark. For $B(x; F)$ under the uniform norm where $X = [a, b]$ and $F = \mathbb{R}$, $B(f, \epsilon)$ is an ϵ-tube around f. Thus,

$$\|f - g\|_u < \epsilon \Rightarrow \sup_{x \in [a,b]} |f(x) - g(x) < \epsilon|.$$

Remark. If f is bounded, we can define $\|f\|_u = +\infty$, even though $f \notin B(X; F)$. Hence

$$\|\cdot\|_u : F^X \to [0, +\infty], \quad B(X; F) = \left\{ f \in F^X : \|f\|_u < +\infty \right\}.$$

In 425a, we knew that every normed vector space can be thought of as a metric space. Hence, we can consider the properties like convergence and continuity.

Proposition. In any normed F-vector space $(V, \|\cdot\|)$, the following functions are continuous:

- Translation: $f : (V, \|\cdot\|) \to (V, \|cdot\|)$ defined by $f(v) = v + v_0$,
- Multiplication by scalars: $g : (V, \|cdot\|) \to (V, \|cdot\|)$ defined by $g(v) = \alpha v$,
- Taking the norm: $h : (V, \|\cdot\|) \to [0, \infty)$ defined by $h(v) = \|v\|$.

In general, f and g are homeomorphisms. They are:

- bijective,
- continuous,
- and they have continuous inverses.

There are useful vector spaces that are *not* $\mathbb{R}^n$, but they are essentially isomorphic to $\mathbb{R}^n$. Consider the following.

Example. Suppose $V \subset B([0,2\pi];\mathbb{R})$,

$$V = \operatorname{span}\left(\{\sin(nx),\cos(nx)\}_{n=1}^{N} \cup \{1\}\right).$$

V has basis $B = (1, \sin x, \cos x, \ldots, \cos Nx, \sin Nx)$; it has dimension $\dim V = 2N+1$.

Proposition. Let V be an n-dimensional F-vector space. Assume $B = (v_1, \ldots, v_n)$ is a basis for V, $\|\cdot\|_V$ is a norm on V, and $L_B : V \to F^n$ is the coordinate map $L_B(c_1 v_1 + \ldots + c_n v_n) = (c_1, \ldots, c_n)$. Then

$$\|\cdot\|_a : F^n \to [0,\infty) = \|(c_1, \ldots, c_n)\|_a = \left\|L_B^{-1}(c_1, \ldots, c_n)\right\|_V$$

is a norm on F^n, and

$$L_B : (V, \|\cdot\|_V) \to (F^n, \|\cdot\|_a)$$

is an isometry and therefore a homeomorphism.

We now define Banach spaces.

Definition. Let $(V, \|\cdot\|)$ be a normed F-vector space. If $(V, \|\cdot\|)$ is complete, then $(V, \|\cdot\|)$ is a Banach space.

Remark. Note that a set is complete if every Cauchy sequence in it converges.

We give some examples. Define a space of functions with a metric space (X, d) and $F = \mathbb{R}$ or $\mathbb{C}$.

Definition.

- $C(X;F)$ is the set of continuous functions $f : X \to F$.
- $BC(X;F)$ is the set of bounded continuous functions. That is, $BC(X;F) = C(X;F) \cap B(X;F)$.
- $C_C(X;F)$ is the set of compactly supported continuous functions on X.

Remark. We define the support of f by

$$\operatorname{supp}(f) = \operatorname{Cl}_X \{x \in X : f(x) \neq 0\}.$$

f is compactly supported if $\operatorname{supp}(f)$ is compact.

Additionally, we always equip subspaces of $B(X;F)$ with $\|\cdot\|_u$. However, this is not the only possible norm.

Remark. $(BC(X;F), \|\cdot\|_u)$ is a Banach space for $F = \mathbb{R}$ or $F = \mathbb{C}$. This can be proven using the completeness of $\mathbb{R}$ and the uniform limit theorem.

Example. The L^p norm is an alternative to the uniform norm. On $C([a,b])$,

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}, \quad p \in [1,\infty).$$

Beginning of February 3, 2023

Last time, we talked about normed vector spaces and function spaces. Sometimes it might be helpful to define more than one norm on the same vector space. In fact, we are able to compare the two norms by equivalency.

Definition. $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, denoted $\|\cdot\|_a \sim \|\cdot\|_b$, if there exists $c, C > 0$ such that for all $v \in V$,

$$c\|v\|_b \leqslant \|v\|_a \leqslant C\|v\|_b.$$

$\|\cdot\|_a$ is strictly weaker than $\|\cdot\|_b$ if $\|\cdot\|_a \nsim \|\cdot\|_b$, but there exists $C > 0$ such that for all $v \in V$,

$$\|v\|_a \leqslant C\|v\|_b.$$

$\|\cdot\|_a$ and $\|\cdot\|_b$ are comparable if they are either equivalent, or one is strictly weaker than the other.

Example. In F^n, the Euclidean norm and the uniform norm are equivalent. Consider $x = (x_1, \ldots, x_n)$. By the uniform norm, $\|x_j\| = \max_{i\in[n]} \|x_i\| = \|x\|_u$. Note that for some $j \in [n]$ where $\|x\|_u$ is defined upon,

$$\|x\|_u = \sqrt{|x_j|^2} \leqslant \sqrt{|x_1|^2 + \ldots + |x_n|^2} = \|x\| \leqslant \sqrt{n|x_j|^2} = \sqrt{n}\|x\|_u.$$

Example. In the space $C^1([a,b])$, consider $\|f\|_{C^1} = \|f\|_u + \|f'\|_u$. We know that $\|f\|_u \leqslant \|f\|_{C^1}$. We **claim** that $\|\cdot\|_{C^1}$ is strictly stronger. It suffices to show that there is no C such that $\|f\|_{C^1} \leqslant C\|f\|_u$. Proving by contradiction, choose $C > 0$ and suppose $\|f\|_{C^1} \leqslant C\|f\|_u$. As the C^1 norm encodes information *not* presented by the uniform norm - the derivative - it now suffices to find a function that oscillates fast yet has small amplitude. Pick $f(x) = \sin(n(x-a))$ and let $n > C$. Then $f'(a) = n > C$, contradicting with the assumption.

Theorem. Let V be an F-vector space, and let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on V. Let $\mathcal{T}_a$ and $\mathcal{T}_b$ denote the associated topologies. The following are equal:

(1) b-norm is stronger than a-norm: $\|v\|_a \leqslant C\|v\|_b$ for some $C > 0$ for all $v \in V$.

(2) $B_a(0,1)$ is open in $(V, \|\cdot\|_b)$.

(3) There exists $r > 0$ such that $B_b(0,r) \subset B_a(0,1)$.

(4) $\mathcal{T}_a \subset \mathcal{T}_b$.

(5) If $(v_i)_{i=1}^\infty$ is a sequence that converges in $(V, \|\cdot\|_a)$, then it also converges in $(V, \|\cdot\|_b)$.

(6) $\mathrm{id} : (V, \|\cdot\|_a) \to (V, \|\cdot\|_b)$ is continuous.

(7) $v \mapsto \|v\|_a$ is continuous on $(V, \|\cdot\|_b) \to [0, \infty)$.

Corollary. $\|\cdot\|_a \sim \|\cdot\|_b$ if and only if $\mathcal{T}_a = \mathcal{T}_b$.

Proof: (1) implies (2). Assume $\|v\|_a \leqslant C\|v\|_b$. Consider an arbitrary point $x \in B_a(0,1)$, we attempt to show that there exists a neighborhood defined on the b-norm that is also included in the a-ball. Consider first $r = 1 - \|x\|_a$, which implies $B_a(x,r) \subset B_a(0,1)$. By assumption,

$$B_a(x,r) = \{y : \|y-x\|_a < r\} \supset \{y : \|y-x\|_b < rC^{-1}\} = B_b(x, rC^{-1}).$$

Thus $B_b(x, rC^{-1})$ is a neighborhood contained in $B_a(0,1)$. As the choice of x is arbitrary, each $x \in B_a(0,1)$ is an interior point under the b-norm, hence $B_a(0,1)$ is open. □

Proof: (2) implies (3). If $B_a(0,1)$ is open in $(V, \|\cdot\|_a)$, considering zero is an interior point of $B_a(0,1)$ with respect to $\|\cdot\|_b$, we are automatically done. □

Proof: (3) implies (1). Suppose $B_b(0,r) \subset B_a(0,1)$, i.e., $\|v\|_b < r \Rightarrow \|v\|_a < 1$. Choose $v \in V \setminus \{0\}$, and define $w = \frac{v}{\|v\|_b}\frac{r}{1+\delta}$ for some $\delta > 0$. As $\delta \to 0$, $w \to r^-$, thus $w \in B_b(0,r) \subset B_a(0,1)$. Knowing that $\|w\|_a < 1$, this implies

$$\frac{\|v\|_a}{\|v\|_b} \cdot \frac{r}{1+\delta} \leqslant 1 \Rightarrow \|v\|_a \leqslant \frac{1+\delta}{r}\|v\|_b .$$

Letting $\delta \to 0$ gives $C = r^{-1}$. □

Proof: (2) implies (4). Assume $B_a(0,1)$ is open in $(V, \|\cdot\|_b)$. From this we know that $B_a(0,1) \in \mathcal{T}_b$, which implies $B_a(x,r) \in \mathcal{T}_b$ (by scalar expansion/contraction and/or translation). As any arbitrary balls in $\mathcal{T}_a$ is also an element of $\mathcal{T}_b$, any arbitrary open sets, in terms of unions of open balls, also belong to $\mathcal{T}_b$. □

Beginning of February 6, 2023

In the last lecture, we showed that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$, and $2 \Rightarrow 4$. Today we prove the other few.

Proof: (4) implies (5). Assume $\mathcal{T}_a \subset \mathcal{T}_b$. Let $(v_i)_{i=1}^{\infty}$ be a sequence that converges to v in $(V, \|\cdot\|_b)$. Let U be an open set in $(V, \|\cdot\|_a)$ that contains v. $U \in \mathcal{T}_a \subset \mathcal{T}_b$. Therefore there exists $N \in \mathbb{N}$ such that $n \geqslant N \Rightarrow v_n \in U$. Since U is an arbitrary set in $\mathcal{T}_a$, we are essentially done.

Proof: (5) implies (6). Here we consider the sequential characterization of continuity. Wanting to show $v_n \to v$ in $(V, \|\cdot\|_b)$ implies $\mathrm{id}(v_n) \to \mathrm{id}(v)$ in $(V, \|\cdot\|_a)$, we are automatically done by (5).

Proof: (6) implies (7). Consider

$$v_{\|\cdot\|_b} \overset{\mathrm{id}}{\mapsto} v_{\|\cdot\|_a} \mapsto \|v\|_a .$$

The following composition $v \mapsto \|v\|_a$ is a composition of continuous functions, hence is continuous.

(7) implies (2). Assume (7) holds; then $f : (V, \|\cdot\|_b) \to [0,\infty)$ defined by $f(v) = \|v\|_a$ is a continuous mapping. Therefore, $B_a(0,1) = f^{-1}([0,1))$ is the pre-image of an open set in $[0,\infty)$, hence is also open. □

We have equipped ourselves with sufficient tools to tackle the problem of norm equivalencies in finite-dimensional real or complex vector spaces. This is our next goal.

Theorem. All norms are equivalent on any finite-dimensional real or complex vector space V.

Proof. It suffices to consider the case $V = F^n$ where $F = \mathbb{R}$ or $\mathbb{C}$. Indeed, suppose this case is proven, consider W be an n-dimensional F-vector space with basis $(w_1, ..., w_n)$. Then let $\|\cdot\|_{W_a}$ and $\|\cdot\|_{W_b}$ be two norms on W. Define norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on F^n by

$$\|(c_1, ..., c_n)\|_a = \|c_1 w_1 + ... + c_n w_n\| W_a, \quad \|(c_1, ..., c_n)\|_b = \|c_1 w_1 + ... + c_n w_n\|_{W_b}.$$

As $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent by assumption, $\|\cdot\|_{W_a}$ and $\|\cdot\|_{W_b}$ are equivalent as well.
Now it is left to prove that any α-norm $\|\cdot\|_\alpha$ on F^n is equivalent to $\|\cdot\|$.

Proof. It suffices to show the existence of $c, C > 0$ such that

$$c\|v\| \overset{(*)}{\leqslant} \|v\|_\alpha \overset{(**)}{\leqslant} C\|v\|_u$$

$(*)$ Let $(e_1, ..., e_n)$ denote the standard basis. Then for $v \in V$,

$$\|v\|_\alpha = \left\|\sum_{i=1}^n v_i e_i\right\|_\alpha \leqslant \sum_{i=1}^n |v|_i \|e_i\|_\alpha \leqslant \|v\|_u \sum_{i=1}^n \|e_i\|_\alpha.$$

As $\sum_{i=1}^n \|e_i\|_\alpha$ is only dependent on α and independent on v, name that C_α for each α and we are done.
$(**)$ Consider the boundary of $n-1$-dimensional sphere: $S^{n-1} = \{v : \|v\| = 1\}$. By Heine-Borel, S^{n-1} is compact. Knowing that the Euclidean norm, equivalent to the uniform norm, is at least stronger than the α-norm, consider $f : (V, \|\cdot\|) \to [0, \infty)$ that maps $v \mapsto \|v\|_\alpha$ is continuous. Thus $f|_{S^{n-1}}$ has a minimum value $c \neq 0$. (Well, if $c = 0$, it wouldn't make sense as only the zero vector has zero norm.) Considering $v \in V \setminus \{0\}$,

$$\frac{v}{\|v\|} \in S^{n-1} \Rightarrow \left\|\frac{v}{\|v\|}\right\|_\alpha \geqslant c \Rightarrow \|v\|_\alpha \geqslant c\|v\|.$$

Then we are essentially done. □

The last part of the class will be devoted to the introduction of multiple integration, as the understanding of elementary measure theory may be helpful to learning the rest of the content.

Definition. We say $Z \subset \mathbb{R}^n$ has measure zero if for every $\epsilon > 0$ there exists a countable collection $\mathcal{A}$ of open n-cells $R_i = (a_{i1}, b_{i1}) \times ... \times (a_{in}, b_{in})$ that cover Z such that

$$\sum_i [\underbrace{\prod_{j=1}^n (b_{ij} - a_{ij})}_{\text{volume of } R_i}] < \epsilon.$$

Beginning of February 8, 2023

Last lecture we discussed the definition of measure zero. Today we will first discuss Riemann integration in $\mathbb{R}^2$, constructing the integral from scratch.
Consider a rectangle $R = [a, b] \times [c, d]$. Recall that in the one-dimensional case we partition $[a, b]$ into n partitions; here we consider

$$P = \{x_i\}_{i=0}^n, \quad a = x_0 < x_1 < ... < x_n = b; \quad Q = \{y_j\}_{j=0}^m, \quad c = y_0 < ... < y_m = d.$$

Therefore we obtain a two-dimensional "grid" $G = P \times Q$, partitioning the rectangle into mn parts. Denote $R_{ij} = I_i \times J_j$, where $I_i = [x_{i-1}, x_i]$, $J_j = [y_{j-1}, y_j]$. The area of each part is then

$$|R_{ij}| = |I_i| \cdot |J_j| = \Delta x_i \cdot \Delta y_j.$$

Additionally, we can choose a sample (s_{ij}, t_{ij}) in each of the rectangles. Multiplying the function value $f(s_{ij}, t_{ij})$ at the sample point with the area of the rectangle R_{ij} gives the volume of the "rectangular prism" at the ij-th part. Therefore it makes sense to define the Riemann sum as following:

$$R(f, G, S) = \sum_{i,j} f(s_{ij}, t_{ij}) |R_{ij}|, \qquad \begin{cases} U(f, G) = \sup_S R(f, G, S) = \sum_{i,j} M_{ij} |R_{ij}| \\ L(f, G) = \inf_S R(f, G, S) = \sum_{i,j} m_{ij} |R_{ij}|, \end{cases}$$

Here M_{ij} and m_{ij} is the supremum and infimum of f on the ij-th minor rectangle. The upper Riemann integral is defined as the infimum of $U(f, G)$ over all choices of partitioning; the lower Riemann integral is defined as the supremum of $L(f, G)$ over all choices of partitioning. The function f is Riemann integrable on R if the upper and lower Riemann sums equate: if for every $\varepsilon > 0$ there exists a partition G such that $U(f, G) - L(f, G) < \varepsilon$. This part is identical to the one-dimensional case.

Consider the following theorem (which may be surprising,) which we will prove later.

Theorem. (*Riemann-Lebesgue*) A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if its set of discontinuities has measure zero.

We don't have the proper tools to prove the theorem. We will leave it aside and come back to it when we do.

Proposition. In $\mathbb{R}^n$, the following sets have measure zero (to be proven as an exercise):

- Any finite set
- Any finite or countable union of measure zero sets
- Any subset of a measure zero set
- $\mathbb{R}^{n-1} \times \{0\}$
- The image of a measure zero set under a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$.

Definition. Define $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$. Given $x \in U$, the oscillation of f at x is defined by

$$\operatorname{osc}_x(f) = \lim_{r \to 0} \left(\sup_{y \in B(x,r)} f(y) - \inf_{z \in B(x,r)} f(z) \right)$$

Example. Consider $f(x) = \sin(x^{-1})$ for $x \neq 0$ and 0 when $x = 0$. Here

$$\operatorname{osc}_0(f) = 1 - (-1) = 2.$$

Example. For a function $g \in \mathbb{R} \to \mathbb{R}$ with a jump discontinuity at $x = c$ (assume $g(c+) > g(c-)$),

$$\begin{cases} \sup_{y \in B(x,r)} g(y) = g(c+) + \varepsilon(r) \\ \inf_{z \in B(x,r)} g(z) = g(c-) - \gamma(r) \end{cases} \Rightarrow \operatorname{osc}_c(g) = g(c+) - g(c-).$$

Proposition. f is continuous at x if and only if $\operatorname{osc}_x(f) = 0$.

The proposition is rather trivial in proof; however, it has great implications. Particularly, we can consider

$$D = \text{discontinuities of } f := \bigcup_{k=1}^{\infty} \left\{ x : \operatorname{osc}_x(f) > \frac{1}{k} \right\}.$$

Proof of Riemann-Lebesgue Theorem. ($\Rightarrow$) Assume f is Riemann integrable. It suffices to prove that D_k' has measure zero for all $k \in \mathbb{N}$, where

$$D_k' = D_k \backslash \{\text{gridlines of partition}\}, \quad D = \bigcup_{k=1}^{\infty} D_k.$$

The rationale behind which is that if D_k' has measure zero, D_k also has measure zero as the gridlines have measure zero from the previous proposition. Subsequently, D, as the countable union of D_k, has measure zero. (Out of time. More in next lecture...)

Beginning of February 10, 2023

Today we continue with the unfinished proof on the Riemann-Lebesgue theorem.

Theorem. (*Riemann-Lebesgue*) A bounded function $f : R \to \mathbb{R}$ is Riemann integrable on $\mathbb{R}$ if and only if its set D of discontinuities has measure zero.

Proof of the two-dimensional case. ($\Rightarrow$) Define $D_k = \left\{(x, y) \in R : \operatorname{osc}_{(x,y)}(f) \geqslant \frac{1}{k}\right\}$. It suffices to show that D_k has measure zero for every $k \in \mathbb{N}$. The main idea is to cover D_k with open rectangles associates to some sufficiently fine grid $G = P \times Q$. For D_k, define $D_k' = D_k \backslash [(P \times \mathbb{R}) \cup (\mathbb{R} \times Q)]$, i.e., D_k' is D_k minus the "grid lines" of G. Consider the collection of open rectangles $\{R_{ij}\}$ associated to grid G. Consider

$$\mathcal{B} := \{R_{ij} \text{ which intersect } D_k'\} \supset D_k'.$$

It then suffices to show that $\mathcal{B}$ has infinitesimal hypervolume. Consider

$$\sum_{R_{ij} \in \mathcal{B}} |R_{ij}| \leqslant k \sum_{R_{ij} \in \mathcal{B}} (M_{ij} - m_{ij}) |R_{ij}| = k(U(f, G) - L(f, G)) < \varepsilon,$$

last step by the fact that f is Riemann integrable. **Note** that $M_{ij} = \sup_{x \in \mathrm{Cl}(R_{ij})} f(x)$, $m_{ij} = \inf_{x \in \mathrm{Cl}(R_{ij})} f(x)$.
($\Leftarrow$) Define $D_k = \left\{(x, y) \in R : \operatorname{osc}_{(x,y)} f \geqslant \frac{1}{k}\right\}$. Suppose D has measure zero. Given a grid G and $k \in \mathbb{N}$, each R_{ij} belongs to one of the following:

$$\mathcal{G} = \left\{ R_{ij} : M_{ij} - m_{ij} < \frac{1}{k} \right\}, \quad \mathcal{B} = \left\{ R_{ij} : M_{ij} - m_{ij} \geqslant \frac{1}{k} \right\}.$$

Consider the difference between upper and lower Riemann integrals:

$$U(f, G) - L(f, G) = \sum_{R_{ij} \in \mathcal{G}}^{\infty} (M_{ij} - m_{ij}) |R_{ij}| + \sum_{R_{ij} \in \mathcal{B}} (M_{ij} - m_{ij}) |R_{ij}|.$$

Concerning the blue part, we know that $M_{ij} - m_{ij} < \frac{1}{k}$, hence

$$\sum_{R_{ij}\in\mathcal{G}} (M_{ij} - m_{ij})\,|R_{ij}| < \frac{\sum_{R_{ij}} |R_{ij}|}{k}.$$

Regarding the red part, with $M_{ij} - m_{ij} \leqslant 2\,\|f\|_u$, we have

$$\sum_{R_{ij}\in\mathcal{B}} (M_{ij} - m_{ij})\,|R_{ij}| \leqslant 2\,\|f\|_u \sum_{R_{ij}\in\mathcal{B}} |R_{ij}|\,.$$

With $\sum_{R_{ij}} |R_{ij}|$ fixed, we can first choose $k \in \mathbb{N}$, independent of the grid G chosen, such that $\frac{|R|}{k} < \frac{\varepsilon}{2}$. Here we utilize the fact that $D_k \subset D$ has measure zero for all $k \in \mathbb{N}$. Denote the cover of open rectangles $\{S_k\}_{k=1}^{\infty} \supset D_k$ with total volume less than $\frac{\varepsilon}{4\|f\|_u}$. Here we need an additional lemma: the Lebesgue number lemma.

Lemma. (*Lebesgue*) Let K be a compact subset of a metric space (X, d). Let $\mathcal{U}$ be any covering of K, then there exists a Lebesgue number $\lambda > 0$ such that $E \subset K$ and $\operatorname{diam}(E) < \lambda$ implies $E \subset U$ for some $U \in \mathcal{U}$.

Note that we cannot directly apply Lebesgue number lemma to D_k as D_k may not be compact. However, we can apply similar logic to R. For every $z \in R\backslash D_k$, let W_z be a neighborhood of z such that $\sup_{W_z} f - \inf_{W_z} f \leqslant \frac{1}{k}$. Then $\mathcal{U} = \{W_z\}_{z\in R\backslash D_k} \cup \{S_k\}_{k=1}^{\infty}$ covers R. As R is compact, let λ be the Lebesgue number associated with this covering $\mathcal{U}$, we can choose G such that $\operatorname{diam}(R_{ij}) < \lambda$ for all R_{ij}. As each R_{ij} must in contained in some $U \in \mathcal{U}$ regardless if $R_{ij} \in \mathcal{G}$ or $R_{ij} \in \mathcal{B}$, picking $R_{ij} \in \mathcal{B}$ ensures R_{ij} be contained in some S_l. As each $R_{ij} \in \mathcal{B}$ is contained in some S_k, $\sum_{R_{ij}\in\mathcal{B}} |R_{ij}| \leqslant \sum_{k=1}^{\infty} |S_k| < \frac{\varepsilon}{4\|f\|_u}$. Therefore $U(f, G) - L(f, G) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and we are done. ☐

Based on the Riemann-Lebesgue theorem, we can define the following:

Definition. Let $S \subset \mathbb{R}^n$ be such that ∂S has measure zero. We can define

$$\int_S f = \int_R f \mathbb{I}_S,$$

where R is any rectangle that contains S.

Beginning of February 13, 2023

Today we will cover Fubini's theorem, similar to the one that is covered in multivariable calculus.

Theorem. (*Fubini*) Consider $R = [a, b] \times [c, d]$. Then for continuous f,

$$\iint_R f = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

As we learned previously from Riemann-Lebesgue, we know that f is Riemann integrable if the set of discontinuities has measure zero. We can find some examples where the Fubini's theorem in Calculus 3 fails to work.

Example. Consider $f : [-1,1]^2 \to \mathbb{R}$ defined by $f(x,y) = 1$ for $x \in \mathbb{Q}$, $y = 0$ and $f(x,y) = 0$ otherwise. Note

$$\int_{-1}^{1} f(x,0)\,\mathrm{d}x$$

does *not* make any sense. However, the discontinuities $D = [-1,1] \times \{0\}$ has measure zero, hence the function is indeed Riemann integrable.

With this said, consider the Fubini's theorem · kai, 425b exclusive (actually not.)

Theorem. (*Fubini · kai*) Assume $f : R \to \mathbb{R}$ is bounded. Define the lower and upper slice integrals as

$$\underline{F}(y) = \underline{\int_a^b} f(x,y)\,\mathrm{d}x, \quad \overline{F}(y) = \overline{\int_a^b} f(x,y)\,\mathrm{d}x.$$

If $f \in \mathcal{R}(R)$, then $\underline{F}, \overline{F} \in \mathcal{R}([c,d])$, and

$$\int_R f = \int_c^d \underline{F}(y)\,\mathrm{d}y = \int_c^d \overline{F}(y)\,\mathrm{d}y.$$

Corollary. If $\overline{F} = \underline{F}$ on $[c,d]$, then the Calculus 3 version holds.

Remark. We may (or may not) be able to prove Fubini's theorem using the interchangability of double sums...

Proof of Fubini's theorem. Consider the first statement: $\underline{F}, \overline{F} \in \mathcal{R}([c,d])$. Usually we were asked to create a partition... But right now, as we need to construct our partition given some grid partition R. Hence, choose $\varepsilon > 0$, and let $G = P \times Q$ be a grid such that $U(f,G) - L(f,G) < \varepsilon$. It suffices to prove

$$L(f,G) \overset{(*)}{\leqslant} L(\underline{F},Q) \leqslant U(\overline{F},Q) \overset{(**)}{\leqslant} U(f,G).$$

We only focus on proving $(*)$; $(**)$ can be proven in an analogous way. To this end, consider

$$L(f,G) = \sum_{i,j} m_{ij}\,|R_{ij}| = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} m_{ij}\Delta x_i\right)\Delta y_j, \quad m_{ij} = \inf_{(x,y)\in R_{ij}} f(x,y),$$

$$L(\underline{F},Q) = \sum_{j=1}^{n} m_j \Delta y_j, \quad m_j = \inf_{y\in J_j} \underline{F}.$$

Here m_{ij} is the infimum of f over each grid m_{ij}, and m_j is the infimum of $\underline{F}$ over the strip J_j. Now it suffices to show that, for all j,

$$\sum_{i=1}^{m} m_{ij}\Delta x_i \leqslant m_j.$$

Fix $y = y_0$. By the definition of infimum we have

$$m_{ij} = \inf_{(x,y)\in R_{ij}} f(x,y) \leqslant \inf_{x\in I_i, y=y_0} f(x,y) = \inf_{x\in I_i} f(x,y_0) = m_i(x,y_0).$$

Here the left-hand side essentially takes the infimum over a superset of the right-hand side, hence the inequality between infimums can be established. As a result of this inequality, we have

$$\sum_{i=1}^{m} m_{ij}\Delta x_i \leqslant \sum_{i=1}^{m} m_i(x,y_0)\Delta x_i = L(f(x,y_0),P) \leqslant \int_a^b f(x,y)\,\mathrm{d}x = \underline{F}(y).$$

Taking the infinum over J_j on both sides returns the desired result. ☐

Lastly we will touch a bit on Schauder basis.

Definition. Let $(V, \|\cdot\|)$ be a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$. The sequence $(v_i)_{i=1}^{\infty}$ in V is a Schauder basis if for every $w \in V$ there exists a unique α_i, $i \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} \alpha_i v_i - w \right\| = 0.$$

The key differentiation between Schauder basis and the Hamel basis is that the former introduces approximation by a limit, meanwhile the latter requires the linear combination to equate with w. Namely, $l^p(\mathbb{N};\mathbb{C})$ has Schauder basis $e_n = (0, \cdots, 0, 1, 0, \cdots)$, but e_n is *not* a Hamel basis.

Beginning of February 15, 2023

Having equipped ourselves with the notion of measure zero, we can continue with a more efficient discussion of cheap L^p spaces. They are *not* inexpensive; they are cheap. Note that the real L^p spaces are defined using the Lebesgue integral instead of the Riemann integral. They are, however, useful enough to justify introducing them in this context.

L^p norms measure size in many different ways. However, they are *not* true norms on spaces like $\mathcal{R}_{\text{loc}}(\mathbb{R})$. The primary reason is degeneracy. Specifically, consider the function that takes $f(x) = 0$ for $x \neq a$ and $x = a$ for $x = a$. It is zero a.e., hence its L^p norm is zero (from a previous exercise and from previous lecture on measure zero sets); however, f is *not* the zero function $f(x) = 0$.

We can fix this problem by introducing a new normed vector space, where the elements are *not* functions. Instead, they are equivalent classes of functions.

Definition. For $f, g \in \mathcal{R}_{\text{loc}}(I)$, I a closed interval in $\mathbb{R}$, we say that $f \sim g$ if $f = g$ a.e. in I.

Introducing new temporary notations on equivalent classes:

$$[f] = \{g \in \mathcal{R}_{\text{loc}}(I), \ \ f \sim g\}.$$

An element $g \in [f]$ is a representative of the equivalence class $[f]$. Additionally, consider the space of all equivalence classes, denoted as

$$\mathcal{R}_{\text{loc}}(I, F)/\sim = \{[f] : f \in \mathcal{R}_{\text{loc}}(I, F)\}.$$

Theorem. $\mathcal{R}_{\text{loc}}(I, F)/\sim$ is a vector space under operations $[f] + [g] = [f + g]$, $\alpha[f] = [\alpha f]$.

Remark. In the context of the Riemann integral, $f = g$ a.e. in I is *not* enough to imply $f \sim g$. Both functions need to be locally Riemann integrable. (A rather classic counterexample is the Dirchlet function, which is zero a.e., but is *not* Riemann integrable.)

Well, one of my peers asked, "why are we learning the 'dumb' way but not using the Lebesgue integral?" To answer the question, the professor quoted some famous mathematician:

I am aware of certain universities in England, where the Lebesgue integral is taught to first year undergrad. I am not aware of any universities anywhere where these first year undergrads learn the Lebesgue integral.

Lebesgue integration is *difficult*, and required lots of "prerequisite" material to be defined rigorously. For now, we can first consider the Riemann integral and perhaps wait a semester or two before we take a measure theory class.

Example. Consider an example of an equivalence class: fractions. With simple arguments like $\frac{2}{4} = \frac{1}{2}$ holding, we can define an equivalence relationship like $(2,4) \sim (1,2)$. However, consider the "dumb-dumb" addition of fractions: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$. Why does this *not* work? The essence of the problem is that under this system of addition, we cannot *freely* choose the representative of the equivalence class. (Try taking $a = 1$, $b = 2$ and $a = 2$, $b = 4$ respectively. We return different answers!)

Proof of the addition axiom. With the fraction example failing, if we need to show $[f] + [g]$ is well-defined, it suffices to show that $f \sim f'$ and $g \sim g'$ implies $f + g \sim f^* + g^*$. Consider

$$f \sim f^* \Leftrightarrow Z_f := \{x : f(x) \neq f^*(x)\} \text{ is a measure zero set,}$$

$$g \sim g^* \Leftrightarrow Z_g := \{x : g(x) \neq f^*(g)\} \text{ is a measure zero set.}$$

Now consider $h = f + g$ and $h^* = f^* + g^*$,

$$Z_h = \{x : h(x) \neq h^*(x)\} \subset Z_f \cup Z_g, \text{ a measure zero set.}$$

(The other vector space axioms are proved on a similar note.) □

Remark. We can easily obtain that $[f] + [g] = [f + g] = [g + f] = [g] + [f]$.

Definition. We define cheap L^p space as

$$L^P_{\mathcal{R}}(I) = \{[f] \in \mathcal{R}_{\text{loc}}(I)/\sim\},$$

such that a representative $f(x) \in [f]$ has

$$\int_I |f(x)|^p \, \mathrm{d}x < +\infty.$$

Remark. It can be verified that taking *any* representative in the equivalence class yields the same result. Namely, $f \sim g \Rightarrow |f(x)|^p \sim |g(x)|^p$, so their integral are also the same.

Definition. The L^p norm in cheap L^p spaces is defined by

$$\|[f]\|_p = \left(\int_I |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.$$

Remark. Later, we simply denote $\|[f]\|_p$ as $\|f\|_p$.

At least, we propose a theorem (that we would *not* prove.)

Theorem. Assume $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ and f has finite L^1 norm. Choose $\varepsilon, \delta > 0$. Then there exists $g \in C_c(\mathbb{R})$ (continuously compactly supported) such that $\|f - g\|_1 < \varepsilon$ and $\|g\|_u \leqslant 4\|f\|_u$. If $\operatorname{supp}(f) \subset (a, b)$, then g can be chosen to satisfy $\operatorname{supp}(g) \subset (a - \delta, b + \delta)$.

This is equivalent of saying if we use the L^1 notion of distance, we can take our favorite L^1 function f, and we can find a compactly supported continuous function that is close to f in the L^1-sense. (This statement still holds true even with Lebesgue integrals.) This is particularly important if we want to consider any L^1 function. We will not officially prove it, but (in the next lecture) we will at least try to convince ourselves that it's true. :)

Beginning of February 22, 2023

Consider the theorem from last week (I am quoting the notes from the last lecture): *if we use the L^1 notion of distance, we can take our favorite L^1 function f, and we can find a compactly supported continuous function that is close to f in the L^1-sense.*

Sketch of proof. Consider the function f. It has to decay for large x values (or else it would have infinity L^1 norm.) We can then truncate the function over some compact interval $[A, B]$:

$$f^* = f \cdot \mathbb{I}_{[A,B]}.$$

This allows us to approximate the integral by a step function on a couple of tiny intervals. We can then choose the value of these steps as, for example, the infimum of the function over each interval (the lower Riemann sum.) We then have the area as such - we should be familiar with it:

$$f^{**} = \sum_{i=1}^{N} m_i \mathbb{I}_{[x_{i-1}, x_i)}.$$

However, the step function we get it *not* continuous. Instead, we replace each of the tiny steps with something that *is* continuous. Suppose we have some step function from x_{j-1} to x_j that looks like a rectangle if we plot them out; we can then approximate it by a isoceles trapezoid defined on $(x_{j-1} - \delta, x_j + \delta)$. Specifically, the graph of the approximation function connects the points $(x_{j-1} - \delta, 0) \to (x_{j-1}, m_j) \to (x_j, m_j) \to (x_j + \delta, 0)$. Connecting the individual graphs gives us a function that satisfies the desired properties. □

Remark. Interpreting the statement topologically, the theorem suggests that $C_C^0(\mathbb{R})$ is dense on $(L^1_{\mathcal{R}}(\mathbb{R}), \|\cdot\|_1)$.

After a small discussion on approximation theory, we now move forward to convolutions. In an essence, *convolution is weighted average*. Recall in calculus when we wanted to find the center of mass, or the weighted average of x-coordinates associated to the density $\rho(x)$:

$$\overline{x} = \underbrace{\frac{1}{\int_a^b \rho(x)\,\mathrm{d}x}}_{\text{mass}} \int_a^b x \underbrace{\rho(x)}_{\text{weight}}\,\mathrm{d}x.$$

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f = \mathbb{I}_{[0,\infty)}$. For this function f, the "sliding average" of f over an $(x - \varepsilon, x + \varepsilon)$ interval is

$$f_\varepsilon(x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(z)\,\mathrm{d}z.$$

The unweighted average graph is one if $x > \varepsilon$, zero if $x < -\varepsilon$, and $f_\varepsilon(x) = \frac{1}{2} + \frac{x}{2\varepsilon}$ if $x \in (-\varepsilon, +\varepsilon)$.

Suppose we want to insist integrating over $\mathbb{R}$, we can do so by considering an indicator function

$$f_\varepsilon(x) = \int_{\mathbb{R}} \left(\frac{1}{2\varepsilon} \mathbb{I}_{[x-\varepsilon, x+\varepsilon]} \{z\} \right) f(z) \, \mathrm{d}z = \int_{\mathbb{R}} \left(\frac{1}{2\varepsilon} \mathbb{I}_{[-\varepsilon,\varepsilon]}(x-z) \right) f(z) \, \mathrm{d}z.$$

Definition. Let $\phi, f \in \mathcal{R}_{\mathrm{loc}}(\mathbb{R})$. Assume $\int_{\mathbb{R}} \phi(y) f(x-y) \, \mathrm{d}y$ converges absolutely for every $x \in \mathbb{R}$. The convolution of ϕ and f, denoted $\phi * f$, is defined by

$$(\phi * f)(x) = \int_{\mathbb{R}} \phi(y) f(x-y) \, \mathrm{d}y.$$

Remark. If the ϕ has L^1 norm of unity, we can actually think about convolution as a weighted average.

We then discuss a bit about the properties of convolutions.

Proposition. Assume $\phi, f, g \in \mathcal{R}_{\mathrm{loc}}(\mathbb{R})$. Assume

$$\int_{\mathbb{R}} \phi(y) f(x-y) \, \mathrm{d}y \text{ and } \int_{\mathbb{R}} \phi(y) g(x-y) \, \mathrm{d}y$$

converge absolutely for every $x \in \mathbb{R}$, then

- (Linearity) The convolution of linear combinations are linear:

$$\phi * (cf + dg) = c(\phi * f) + d(\phi * g).$$

- (Commutativity) $\int_{\mathbb{R}} \phi(x-y) f(y) \, \mathrm{d}y$ converges absolutely for every $x \in \mathbb{R}$, and $\phi * f = f * \phi$.
- If $\phi \in L^1_{\mathcal{R}}(\mathbb{R})$ and f is bounded, then

$$\|\phi * f\|_u \leqslant \|\phi\|_1 \|f\|_u .$$

Proof. The linearity follows directly from the definition of convolution and the linearity of Riemann integrals. Commutativity can be proved by a change in variable operation. Regarding the last inequality,

$$|\phi * f(x)| = \left| \int_{\mathbb{R}} \phi(y) f(x-y) \, \mathrm{d}y \right| \leqslant \|f\|_u \int_{\mathbb{R}} |\phi(y)| \, \mathrm{d}y = \|f\|_u \|\phi\|_1 .$$

$\square$

Theorem. Suppose $\phi, f \in \mathcal{R}_{\mathrm{loc}}(\mathbb{R})$, ϕ has finite L^1 norm, and f is bounded. Then $\phi * f$ exists for all $x \in \mathbb{R}$ and is continuous.

Proof. We want to show that $\phi * f(x) - \phi * f(y)$ is small provided that x and y are close together. Indeed, consider the integral representation of the convolution, we have

$$|\phi * f(x) - \phi * f(y)| = \int_{\mathbb{R}} [\phi(x-z) - \phi(y-z)] f(z) \, \mathrm{d}z \leqslant \|f\|_u \int_{\mathbb{R}} |\phi(x-z) - \phi(y-z)| \, \mathrm{d}z.$$

Let's assume for now that $\phi \in C^0_C(\mathbb{R})$, which implies ϕ is uniformly continuous. Choose $r > 0$ large enough such that $\operatorname{supp}(\phi) \subset [-r, r]$. Then $\phi(x-z)$, which is ϕ shifted and flipped, is actually defined on $[x-r, x+r]$. Similarly, $\phi(y-z)$ is defined on $[y-r, y+r]$. Specifically, if $|x-y| < 1$, then

$$\int_{\mathbb{R}} |\phi(x-z) - \Phi(y-z)| \, \mathrm{d}z = \int_{x-r-1}^{x+r+1} |\phi(x-z) - \phi(y-z)| \, \mathrm{d}z.$$

To this end, we can use the fact that ϕ is uniformly continuous to choose $\delta > 0$ such that

$$|x^* - y^*| < \delta \Rightarrow |\phi(x^*) - \phi(y^*)| < \frac{\varepsilon}{\|f\|_u \cdot 2(r+1)}.$$

We now get rid of the assumption of $\phi \in C_C^0(\mathbb{R})$. Let $(\phi)_{n=1}^{\infty}$ be a sequence in $C_C(\mathbb{R})$ such that $\|\phi - \phi\|_1 \to 0$. By the previous proof we know that $\phi * f$ is continuous for all $n \in \mathbb{N}$. Indeed, consider

$$\|\phi * f - \phi * f\|_u = \|(\phi - \phi) * f\|_u \leqslant \|\phi - \phi\|_1 \|f\|_u \to 0.$$

Thus $\phi \rightrightarrows \phi$, and applying the uniform limit theorem completes the proof. □

Beginning of February 24, 2023

Last time, we went over the definition and some properties of convolution. Namely,

$$\phi * f(x) := \int_{\mathbb{R}} \phi(x-y) f(y) \, \mathrm{d}y = \int_{\mathbb{R}} \phi(y) f(x-y) \, \mathrm{d}y = f * \phi(x).$$

Additionally, if $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ and bounded, and $\phi \in L^1_{\mathcal{R}}(\mathbb{R})$, then $\phi * f$ is continuous.
Convolutions can be thought of as weighted averages; the above theorem of continuity helps us approximating "bad" functions f by nicer ones $\phi * f$, if ϕ has certain properties. This brings up the topic of approximate identities. Recall that if $\phi^\varepsilon = \frac{1}{2\varepsilon}\mathbb{I}_{[-\varepsilon,\varepsilon]}$ and $f = \mathbb{I}_{[0,\infty)}$, we have

$$\phi^\varepsilon * f(x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(y) \, \mathrm{d}y.$$

If we wish to consider the average over some neighborhood, the average would be zero if the neighborhood lies entirely left of $-\varepsilon$, one if the neighborhood lies entirely right of ε, and variable otherwise. Note that as ε gets smaller, the function ϕ^ε gets closer to f. This brings up the concept of approximate identity.

Definition. Let $(\phi_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{R}_{\text{loc}}(\mathbb{R})$. We say that $(\phi_n)_{n=1}^{\infty}$ is an approximate identity if

- $\int_{\mathbb{R}} \phi_n(x) \, \mathrm{d}x = 1$ for all $n \in \mathbb{N}$.
- (If $\phi_n < 0$ for some $x \in \mathbb{R}$) $\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} |\phi_n(x)| \, \mathrm{d}x < +\infty$.
- For every $\delta > 0$, $\lim_{n\to\infty} \left[\int_{-\infty}^{-\delta} |\phi_n(x)| \, \mathrm{d}x + \int_{\delta}^{\infty} |\phi_n(x)| \, \mathrm{d}x\right] = 0$.

Remark. The approximate identity is a sequence of functions, not a single function.

Example. The sequence of functions $\phi_n = \frac{n}{2}\mathbb{I}_{[-\frac{1}{n},\frac{1}{n}]}$ is an approximate identity.

Theorem. Let $(\phi_n)_{n=1}^{\infty}$ be an approximate identity on $\mathbb{R}$. Suppose $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ and bounded.

- If f is continuous at x, then $\phi_n * f(x) \to f(x)$.
- If f is continuous on (a,b), then $\phi_n * f(x) \rightrightarrows f$ on $[c,d] \subset (a,b)$.
- (Will not be proven) $\phi_n * f \to f$ in L^p for any $p \in [1,\infty)$.

Proof of convergence. It suffices to show that $\phi_n * f(x) - f(x) \overset{n\to\infty}{\to} 0$. Considering the integral representation,

$$\phi_n * f(x) - f(x) = \int_{\mathbb{R}} \phi_n(y) f(x-y)\,\mathrm{d}y - \int_{\mathbb{R}} f(x)\phi_n(y)\,\mathrm{d}y = \int_{\mathbb{R}} \phi_n(y)\left[f(x-y) - f(x)\right]\,\mathrm{d}y.$$

As we wish to use the third identity in the definition of approximate identity, we split the integral into three:

$$I = \int_{-\infty}^{-\delta} \phi_n(y)\left[f(x-y)-f(x)\right]\,\mathrm{d}y + \int_{\delta}^{\infty} \phi_n(y)\left[f(x-y)-f(x)\right]\,\mathrm{d}y + \int_{-\delta}^{\delta} \phi_n(y)\left[f(x-y)-f(x)\right]\,\mathrm{d}y.$$

Each of the three integrals are small, but of different reasons. Starting from $\int_{-\delta}^{\delta}\,\mathrm{d}y$, choose $\delta > 0$ such that

$$|y| < \delta \Rightarrow |f(x-y) - f(x)| \leqslant \frac{\varepsilon}{2M},$$

where M is the upper bound for $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} |\phi_n(x)|\,\mathrm{d}x$. This way,

$$\left|\int_{-\delta}^{\delta} \phi_n(y)\left[f(x-y)-f(x)\right]\,\mathrm{d}y\right| \leqslant \int_{-\delta}^{\delta} |\phi_n(y)| \underbrace{|f(x-y)-f(x)|}_{<\frac{\varepsilon}{2M}}\,\mathrm{d}y < \frac{\varepsilon}{2M} \times M = \frac{\varepsilon}{2}.$$

For $\int_{-\infty}^{-\delta}\,\mathrm{d}y$ and $\int_{\delta}^{\infty}\,\mathrm{d}y$, we use the fact that f is bounded (hence it has finite uniform norm):

$$\left|\int_{-\infty}^{-\delta} \phi_n(y) \underbrace{\left[f(x-y)-f(x)\right]}_{<2\|f\|_u}\,\mathrm{d}y + \int_{\delta}^{\infty} \phi_n(y) \underbrace{\left[f(x-y)-f(x)\right]}_{<2\|f\|_u}\,\mathrm{d}y\right| \leqslant 2\,\|f\|_u \left[\int_{-\infty}^{-\delta} \phi_n(y)\,\mathrm{d}y + \int_{\delta}^{\infty} \phi_n(y)\,\mathrm{d}y\right].$$

Now we can choose n large enough such that the integrals of the right-hand side sums up to less than $\frac{\varepsilon}{4\|f\|_u}$. Therefore for any $\varepsilon > 0$ we have $|\phi_n * f(x) - f(x)| < \varepsilon$, which completes the proof for pointwise convergence at each x where f is continuous. To prove for uniform convergence, we use the fact that f is uniformly continuous on $[c,d] \subset (a,b)$. In fact, we choose a $\delta > 0$ on a wider interval of $[c-\delta, d+\delta]$ to prevent the case where $x-y$ goes "out of the range". Thus for this δ, $|f(x-y) - f(x)| < \frac{\varepsilon}{2M}$ for $|y| < \delta$, and we are done by running the entire argument. $\square$

Beginning of February 27, 2023

Last time we talked over approximate identities, which allows us to estimate a L^1-finite function by not only a continuous function but also smooth function. Today we will continue on the theory of approximation. Namely, we wish to approximate a continuous function $f : [a,b] \to \mathbb{R}$ uniformly by a polynomial. We need a trick up our sleeves:

$$\int_a^b (x-y)^n f(y)\,\mathrm{d}y = \sum_{k=0}^{n} \binom{n}{k} x^k \int_a^b (-y)^{n-k} f(y)\,\mathrm{d}y$$

is a polynomial in x. A natural extension is that

$$\int_a^b p(x-y) f(y)\,\mathrm{d}y$$

is a polynomial in x if p is. (The latter equation is a linear combination of the $\int_a^b (x-y)^n f(y)\,\mathrm{d}y$ for some n's.) The equations above look similar to convolutions - they indeed *are*. Specifically, for some $r > b-a$,

$$\int_a^b p(x-y) f(y)\,\mathrm{d}y = p * \left(f\mathbb{I}_{[a,b]}(x)\right) = \left(p\mathbb{I}_{[-r,r]}\right) * \left(f\mathbb{I}_{[a,b]}\right).$$

The question now brings down to: *can I build an approximation identity of the form $\phi_n = \mathbb{I}_{[-r,r]} \cdot p_n$?*

WLOG, consider $r = 1$. (Why?) We start with an arbitrary function to start with:

$$\phi_1(x) = (1 - x^2)c^{-1}\mathbb{I}_{[-1,1]}, \quad c_1 = \int_{-1}^{1} (1 - x^2)\, \mathrm{d}x.$$

To center the mass towards the center, we simply take the power to the term $(1 - x^2)$ to obtain

$$\phi_n(x) = (1 - x^2)^n c_n^{-1}\mathbb{I}_{[-1,1]}, \quad c_n = \int_{-1}^{1} (1 - x^2)^n\, \mathrm{d}x.$$

Lemma. ϕ_n is an approximate identity.

Proof of lemma. By definition, $\|\phi_n\|_1 = 1$; and all ϕ_n are nonnegative. Now as δ_n is an even function that takes zero value outside of $[-1, 1]$, it suffices to show, for $\delta \in (0, 1)$,

$$\lim_{n\to\infty} \int_{\delta}^{1} \phi_n(x)\, \mathrm{d}x = 0.$$

Take $x \in [\delta, 1]$. Consider the quotient that defines $\phi_n(x)$:

$$\phi_n(x) = \frac{(1 - x^2)^n}{2\int_0^1 (1 - x^2)^n\, \mathrm{d}x}.$$

The above part has function is bounded above by $1 - \delta^2$. For the bottom integral, the following inequality holds:

$$(1 - x^2)^n \geqslant \max\left\{1 - nx^2, 0\right\} = (1 - nx^2)\mathbb{I}_{[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}]}.$$

Thus

$$\int_0^1 (1 - x^2)^n\, \mathrm{d}x \geqslant \int_0^{\frac{1}{\sqrt{n}}} (1 - nx^2)\, \mathrm{d}x = \frac{2}{3\sqrt{n}}.$$

Therefore the quotient is now bounded above by

$$\frac{(1 - x^2)^n}{2\int_0^1 (1 - x^2)^n\, \mathrm{d}x} \leqslant \frac{3\sqrt{n}}{4}(1 - \delta^2)^n.$$

Concerning the term $(1 - \delta^2)^n$, we have, by the fact that $(1 + a)^n \geqslant 1 + na$,

$$(1 - \delta^2)^n = \left(\frac{1 - \delta^2}{1 - \delta^2 + \delta^2}\right)^n = \frac{1}{\left(1 + \frac{\delta^2}{1-\delta^2}\right)^n} \leqslant \frac{1}{1 + n\frac{\delta^2}{1-\delta^2}} \Rightarrow \frac{3}{4}\frac{\sqrt{n}}{1 + n\delta^2/(1 - \delta^2)} \overset{n\to\infty}{\to} 0.$$

$\square$

Theorem. (*Weierstraß*) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. here exists a sequence of polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n \rightrightarrows f$ on $[a, b]$.

Proof. WLOG, let $[a, b] = [0, 1]$, and $f \equiv 0$ on $(-\infty, 0] \cup [1, +\infty)$. Let $(\phi_n)_{n=1}^{\infty}$ be as in the previous lemma. Then $\phi_n * f \rightrightarrows$ on $[0, 1]$, and $\phi_n * f$ is a polynomial for each n. Namely, for $x \in [0, 1]$,

$$\phi_n * f(x) = \int_0^1 c_n^{-1}\mathbb{I}_{[-1,1]}(x - y)\left(1 - (x - y)^2\right)^n f(y)\, \mathrm{d}y.$$

$\square$

Lastly we will touch a bit on the Stone-Weierstraß theorem. Proof not required, as it's quite long.

The motivation originates that we want to approximate $f \in C([a, b])$ with elements of some collection of functions $\mathcal{A}$ (an *algebra* - definition to be provided, but the class of polynomials is an algebra) What can $\mathcal{A}$ be?

Particularly, $\mathcal{A}$ cannot "vanish" at any point.

Definition. $\mathcal{A}$ vanishes at $x_0 \in [a,b]$ if $f(x_0) = 0$ for all $f \in \mathcal{A}$.

If $\mathcal{A}$ vanishes at some x_0, then they cannot approximate a function f that is nonzero on x_0.
Additionally, $\mathcal{A}$ must separate points.

Definition. $\mathcal{A}$ separates points if for every $x, y \in [a,b]$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

If $\mathcal{A}$ does not separate points, they they cannot approximate a function f whose $f(x) \neq f(y)$.

Theorem. (*Stone-Weierstraaß*) (For next class.)

Beginning of March 1, 2023

Today we will talk about Stone-Weierstraß theorem. We will state the real-valued Stone-Weierstraß theorem, then we will use that to prove the general case in the complex plane.

Theorem. (*Weierstraß*). Polynomials on $[a,b]$ are dense in $(C([a,b], \|\cdot\|_u)$.

Recall the following definitions.

Definition. Let E be a set, and $\mathcal{A} \subset F^E$ where $F = \mathbb{R}$ or $\mathbb{C}$.

- $\mathcal{A}$ **separates points** on E if for every $x_1, x_2 \in E$ there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.
- $\mathcal{A}$ **vanishes at no point** of E if for every $x_0 \in E$ there exists $f \in \mathcal{A}$ such that $f(x_0) \neq 0$.

Definition. Let E be a set, and $\mathcal{A} \subset F^E$ where $F = \mathbb{R}$ or $\mathbb{C}$. We say that $\mathcal{A}$ is an F-algebra of functions on E if $\mathcal{A}$ is closed under addition and multiplication. Specifically,

$$f, g \in \mathcal{A}, c, d \in F \Rightarrow cf + dg \in \mathcal{A}, fg \in \mathcal{A}.$$

Theorem. (*Stone-Weierstraß*) Let K be a compact metric space. Let $\mathcal{A} \subset \mathbb{R}^K$ be a real algebra. Assume $\mathcal{A}$ separates points on K, and $\mathcal{A}$ vanishes at no point of K. Then $\mathcal{A}$ is dense in $(C(K, \mathbb{R}), \|\cdot\|_u)$. That is, given $f \in C(K, \mathbb{R})$ and $\varepsilon > 0$, there exists $g \in \mathcal{A}$ such that $\|f - g\|_u < \varepsilon$.

We will not prove it in class - it was an hour-long proof. Assuming the real version is true, we will prove an analogous one for complex-valued functions. However, we need one more assumption.

Definition. A complex algebra $\mathcal{A}$ is **self-adjoint** if $f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}$.

Theorem. (*Stone-Weierstraß, complex version*) Let K be a compact metric space, let $\mathcal{A} \subset \mathbb{C}^K$ be a complex algebra. Assume $\mathcal{A}$ separates points on K, and $\mathcal{A}$ vanishes at no point of K, and $\mathcal{A}$ is self-adjoint. Then $\mathcal{A}$ is dense in $(C(K,\mathbb{C}), \|\cdot\|_u)$.

Proof. Consider $\mathcal{A}_\mathbb{R} = \{f \in \mathcal{A} : \mathrm{Im}(f) \subset \mathbb{R}\}$. (Notation abusing alert: $\mathrm{Im}(f)$ is the image of f.) $\mathcal{A}_\mathbb{R}$ is an algebra:

$$f \in \mathcal{A} \Rightarrow \mathrm{Re}(f) = \frac{f + \overline{f}}{2} \in \mathcal{A}_\mathbb{R}.$$

$\mathcal{A}$ vanishes at no points on K; indeed, consider $g(x) = \mathrm{Re}(f(x) \cdot f(x))$, which has nonzero reals. Additionally, $\mathcal{A}$ separates points. Given $x_1, x_2 \in K$ such that $x_1 \neq x_2$, choose $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$, then choose $h \in \mathcal{A}$ such that $h(x_2) \neq 0$. Construct (a random function)

$$f(x) = \frac{h(x)}{h(x_1)}\left[\frac{g(x)}{g(x_1) - g(x_2)} - \frac{g(x_2)}{g(x_1) - g(x_2)}\right].$$

By Stone-Weierstraß, $\mathcal{A}_\mathbb{R}$ is dense in $(C(K, \mathbb{R}, \|\cdot\|_u)$. Thus, given $f \in \mathcal{A}$ write $f = u + iv$ where $u, v \in C(K, \mathbb{R})$. Then choose u_ε, $v_\varepsilon \in \mathcal{A}_\mathbb{R}$ such that

$$\|u - u_\varepsilon\|_u < \frac{\varepsilon}{2}, \quad \|v - v_\varepsilon\|_u < \frac{\varepsilon}{2}.$$

Therefore

$$\|f - (u_\varepsilon + iv_\varepsilon)\|_u < \varepsilon,$$

which finishes the proof. □

Definition. A **trigonometric polynomial** is a function $p : \mathbb{R} \to \mathbb{C}$ of the form

$$p(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}, \quad N \in \mathbb{N}_0, c_n \in \mathbb{C}, \theta \in \mathbb{R}.$$

Remark. Trigonometric polynomials can always be written in the form

$$p(\theta) = a_0 + \sum_{n=1}^{N} a_n \cos n\theta + \sum_{n=1}^{N} b_n \sin n\theta, \quad a_n, b_n \in \mathbb{C}.$$

Denote

$$P_{\text{trig}}([-\pi,\pi]; F) = \text{ trig polynomials on the field } F = \mathbb{R} \text{ or } \mathbb{C},$$

$$C_{\text{per}}([-\pi,\pi]; F) = F\text{-valued } 2\pi\text{-periodic functions defined on } \mathbb{R}.$$

Claim. By Stone-Weierstraß, $P_{\text{trig}}([-\pi,\pi]; F)$ is dense in $(C_{\text{per}}[-\pi,\pi]; F), \|\cdot\|_u)$.
Note that we cannot directly use Stone-Weierstraßdirectly because the domain $\mathbb{R}$ is not compact, and $P_{\text{trig}}([-\pi,\pi]; F)$ does not separate points in $\mathbb{R}$. More in the next lecture.

Beginning of March 3, 2023

Continuing on last lecture, we make one further notation:

$$F_{\text{per}}^{[-\pi,\pi]} = \left\{f \in F^\mathbb{R} : f(\theta + 2\pi) = f(\theta) \ \forall \theta \in \mathbb{R}\right\} = 2\pi\text{-periodic } F\text{-valued functions on } \mathbb{R}.$$

Recall the claim: $P_{\text{trig}}([-\pi,\pi], F)$ is dense in $(C_{\text{per}}([-\pi,\pi], F), \|\cdot\|_u)$. The problem with using Stone-Weierstraßis because the domain $\mathbb{R}$ is not compact, and $P_{\text{trig}}([-\pi,\pi], F)$ does not separate points on $\mathbb{R}$.

To fix the problem, consider the unit circle in $\mathbb{C}$, defined as

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

There exists a bijective mapping from the real interval $[-\pi, \pi)$ to the complex unit circle S^1. Thus, consider the function $f \in F^{S^1}$ and the function $\tilde{f} \in F_{\text{per}}^{[-\pi,\pi]}$. The transformation

$$f \overset{\Theta}{\rightsquigarrow} \tilde{f}$$

is a well-defined isometry with respect to the uniform norm. Specifically, $\|f\|_u = \|\tilde{f}\|_u$. Additionally, denote

$$P(S^1, F) = \left\{ p \in F^{S^1}, p(z) = \sum_{n=-N}^{N} c_n z^n \right\}; \quad \Theta\left(P(S^1, F)\right) = P_{\text{trig}}([-\pi, \pi], F).$$

Additionally, $P(S^1, F)$ is dense in $(C(S^1; F); \|\cdot\|_u)$ by Stone-Weierstraß theorem. (To be proven in a homework exercise.) Therefore, P_{trig} is dense in C_{per} by the isometric mapping Θ.

We will then move on to another topic: inner product spaces and best approximation. We will think of the projection of a vector to another vector as the "closest distance" in a certain subspace.

Definition. Suppose V is a vector space; U and W are subspaces of V.

- $U + W = \{u + w \in V : u \in U, w \in W\}$ is the **sum** of U and W in V. It is a subspace of V.
- V is the **direct sum** of U and W, if for every $v \in V$ there exists a unique $u \in U$ and $w \in W$ such that $v = u + w$. This is written as $V = U \oplus W$.

Proposition. $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$.

Proof. ($\Rightarrow$) the fact that $V = U \oplus W$ implies $V = U + W$ is obvious. Additionally, choose $v \in U \cap W$. Then

$$v = v + 0 = 0 + v,$$

then both v and 0 has to bee in both U and W. Then v can only be the zero subspace.

($\Leftarrow$) Assume $V = U + W$ and $U \cap W = \{0\}$. Choose $v \in V$; write

$$v = u_1 + w_1 = u_2 + w_2 \Rightarrow u_1 - u_2 = w_2 - w_1.$$

As $u_1 - u_2 \in U$ and $w_2 - w_1 \in W$, they have to be zero because U and W don't intersect elsewhere. ☐

Consider the following question: given a vector space V and a subspace $U \subset V$, can we find an explicit complementary subspace W such that $V = U \oplus W$? It turns out that if an inner product is defined on the vector space, we can define $W = U^{\perp}$, a complemental subspace to U.

Definition. A **projection operator** $P : V \to V$ is a linear map such that $P^2 = P \circ P = P$.

Theorem. Let V be a vector space, and $P: V \to V$ is a projection.

- For $u \in \operatorname{Im}(P)$, $P(u) = u$;
- $P(I-P) = 0$; $I - P$ is also a projection, $\operatorname{Im}(I-P) = \operatorname{Ker}(P)$.
- $V = \operatorname{Im}(P) \oplus \operatorname{Im}(I-P) = \operatorname{Im}(P) \oplus \operatorname{Ker}(P)$.

Definition. The kernel of a transformation P is defined as

$$\operatorname{Ker}(P) = \{v \in V : Pv = 0\}.$$

Corollary. If $V = U \oplus W$ and we define $P: V \to V$ by $P(u+w) = u$, $u \in U$ and $w \in W$, then P is the unique projection on V such that $\operatorname{Im}(P) = U$ and $\operatorname{Ker}(P) = W$.

Proof of first bullet. If $u \in \operatorname{Im}(P)$, then $u = Pv$ for some $v \in V$. This implies $Pu = P^2 v = Pv = u$.

Proof of second bullet. Consider the transformation $P(I-P)v = Pv - P^2 v = 0$, hence $P(I-P)$ is the zero function. Now consider the transformation $(I-P)^2$:

$$(I-P)(I-P) = I(I-P) - P(I-P) = I - P - 0 = I - P.$$

Consider the last statement on $\operatorname{Im}(I-P) = \operatorname{Ker}(P)$. ($\subset$) we know for a fact that everything that belongs to $\operatorname{Im}(I-P)$ takes some for m of $(I-P)v$; but $P(I-P)v = 0$, so $\operatorname{Im}(I-P) \subset \operatorname{Ker}(P)$. ($\supset$) additionally, suppose $Pu = 0$, we have that $(I-P)u = u$, therefore $\operatorname{Ker}(P) \subset \operatorname{Im}(I-P)$.

Proof of third bullet. Consider

$$I = I - P + P.$$

Therefore for every $v \in V$,

$$v = (I-P)v + Pv.$$

The first element belongs to $\operatorname{Im}(I-P)$, and the second element belongs to $\operatorname{Im}(P)$. Thus

$$V = \operatorname{Im}(P) + \operatorname{Im}(I-P) = \operatorname{Im}(P) + \operatorname{Ker}(P).$$

Additionally, suppose $v \in \operatorname{Im}(P) \cap \operatorname{Ker}(P)$. By the fact that $v \in \operatorname{Im}(P)$, $v = Pv$; but $0 = Pv$ as $v \in \operatorname{Ker}(P)$. Thie completes the proof. □

We will prove the corollary in the next lecture.

Beginning of March 6, 2023

Last time we talked a bit about projection operators: a linear mapping $P: V \to V$ such that $P^2 v = Pv$. An example is the projection to a subspace in $\mathbb{R}^n$. We also showed that $V = \operatorname{Im}(P) \oplus \operatorname{Ker}(P)$. We further claimed that the direct sum decomposition uniquely defines a projection P.

Corollary. If $V = U \oplus W$ and $P : V \to V$ is defined by $P(u+w) = u$ for $u \in U$ and $w \in W$, then P is the unique projection on V such that $\operatorname{Im}(P) = U$ and $\operatorname{Ker}(P) = W$.

Proof. It is easy to show that $\operatorname{Im}(P) = U$; let $w = 0$ and pick whatever $u \in U$ desired. Additionally, for $\operatorname{Ker}(P)$, take $P(0+w) = 0$ gives the desired result of $\operatorname{Ker}(P) = W$. Now we prove the uniqueness of P. Suppose $\tilde{P}$ is another projection such that $\operatorname{Im}(\tilde{P}) = U$ and $\operatorname{Ker}(\tilde{P}) = W$. Then

$$u \in U \Rightarrow \tilde{P}u = u; \quad w \in W \Rightarrow \tilde{P}w = 0;$$

Then by the assumption of linearity, $\tilde{P}(u+w) = \tilde{P}u + \tilde{P}w = u$, the same definition as what we started with. $\square$

Example. The orthogonal projection is a classic example that we should think in mind, but certainly it is not the only projection. For example, we can consider $\mathbb{R}^2 = U \oplus W_a$, where $U = \operatorname{span}\{(0,1)\}$, $W_a = \operatorname{span}\{(1,a)\}$. Consider

$$P_a : \mathbb{R}^2 \to \mathbb{R}^2, \ \ P_a(x,y) = (0, y - ax).$$

It is clear that $\operatorname{Im}(P_a) = U$; clearly $P(0,y) = (0,y)$. Then for $\operatorname{Ker}(P_a)$,

$$(x,y) \in \operatorname{Ker}(P_a) \Rightarrow P_a(x,y) = (0, y-ax) = 0 \Rightarrow y = ax \equiv (x,y) = x(1,a) \in W.$$

$$w \in W \Rightarrow w = (x, ax); P_a(x, ax) = (0, ax - ax) = (0,0) \in \operatorname{Ker}(P_a).$$

Thus $\operatorname{Ker}(P_a) = W_a$.

Definition. A (complex) inner product on a complex vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ s.t.

- $\langle x, y \rangle = \langle y, x \rangle$ for every $x, y \in V$;
- $\langle \lambda x + y, z \rangle = \lambda(x, z) + \langle y, z \rangle$;
- $\langle x, x \rangle = 0$ for every $x \in V - \{0\}$.

The vector space V affiliated with the inner product $\langle \cdot, \cdot \rangle$ is defined as a complex inner product space.

Example. The "dot product" on $\mathbb{R}^n$ is an inner product. Similarly, we can define similarly a complex dot product on $\mathbb{C}^n$ as follows:

$$\langle (z_1, \cdots, z_n), (w_1, \cdots, w_n) \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}.$$

Example. The L^2 inner product on $\mathcal{R}([a,b])/\sim$ is defined as

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} \, \mathrm{d}x.$$

The ℓ^2 inner product on $\ell^2(\mathbb{N}, \mathbb{C})$ is defined as

$$\langle (c_n)_n, (d_n)_n \rangle = \sum_{n=1}^{\infty} c_n \overline{d_n}.$$

Proposition. An inner product space is always a normed vector space, with

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Definition. A complete inner product space is called a **Hilbert space**.

Definition. If $u, v \in V$ and V is an inner product space, we say that u and v are **orthogonal** if

$$\langle u, v\rangle = 0.$$

We denote the orthogonality as $u \perp v$.
Additionally, for a subspace W of V, the **orthogonal complement** of W, denoted $W^\perp$, is defined as

$$W^\perp = \{v \in V : \langle v, w\rangle = 0 \ \forall w \in W\}.$$

A set $S \subset V$ is **orthogonal** if $v, w \in S \Rightarrow v \perp W$; **orthonormal** if it's orthogonal and $v \in S \Rightarrow \|v\| = 1$.

Proposition. $W^\perp$ if a subspace of V, and $W \cap W^\perp = \{0\}$.

Proof. We only prove that $W \cap W^\perp \subset \{0\}$. To this end, pick $w \in W \cap W^\perp$; then $\langle w, w\rangle = 0 \Rightarrow w = 0$. □

Consider the situation where $W + W^\perp = V$, (and by the previous results we have that $V = W \oplus W^\perp$.) We can then make the following definition.

Definition. Let $(V, \langle\cdot,\cdot\rangle)$ be a real or complex inner product space, and W is a subspace of V. Assume $V = W \oplus W^\perp$. Define the **orthogonal projection** onto W: $\mathrm{proj}_W : V \to V$ to be the projection operator such that $\mathrm{Im}(\mathrm{proj}_W) = W$ and $\mathrm{Ker}(\mathrm{proj}_W) = W^\perp$.

Theorem. Let $(V, \langle\cdot,\cdot\rangle)$ be a real or complex inner produce space. Take $u \in V - \{0\}$. Then

$$V = (\mathrm{span}\,\{u\}) \oplus (\mathrm{span}\,\{u\})^\perp,$$

and for every $v \in V$,

$$\mathrm{proj}_u(v) = \frac{\langle v, u\rangle}{\|u\|^2} u.$$

Proof. We wish to write $v = cu + w$, where $cu \in (\mathrm{span}\,\{u\})$ and $w \in (\mathrm{span}\,\{u\})^\perp$. Then

$$v - cu = w \in (\mathrm{span}\,\{u\})^\perp \Rightarrow \langle v - cu, u\rangle = 0.$$

From this, we have that $\langle v, u\rangle - c\,\|u\|^2 = 0$, which determines a unique c as

$$c = \frac{\langle v, u\rangle}{\|u\|^2}.$$

By this, we have that

$$v = \frac{\langle v, u \rangle}{\|u\|^2} u + \left(v - \frac{\langle v, u \rangle}{\|u\|^2} u \right),$$

with the first element belonging to $\operatorname{span}\{u\}$ and the second element belonging to $(\operatorname{span}\{u\})^\perp$. Then

$$\operatorname{proj}_u(v) = \operatorname{proj}_u(\cdots) = \frac{\langle v, u \rangle}{\|u\|^2} u.$$

$\square$

Beginning of March 8, 2023

Last time, we discussed about real or complex inner product space $(V, \langle \cdot, \cdot \rangle)$. Particularly, if W is a subspace of V and if $V = W \oplus W^\perp$, then $\operatorname{proj}_W : V \to V$ is the projction on V with image W and kernel $W^\perp$. Additionally,

$$\operatorname{proj}_u v = \frac{\langle v, u \rangle}{\|u\|^2} u. \quad (\|u\| = \sqrt{\langle u, u \rangle})$$

Remark. $V = W \oplus W^\perp$ does *not* work all the time. For example, we can consider $V = C([a, b])$ with

$$\langle f, g \rangle = \int_a^b f \overline{g} \, \mathrm{d}x,$$

where W is the set of all polynomials.

Proposition.

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\operatorname{Re} \langle u, v \rangle.$$

Specifically, $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ if $u \perp v$; this is the general form of Pythagorean theorem.

Proposition. (*Cauchy-Schwarz*)

$$|\langle u, v \rangle| \leqslant \|u\| \, \|v\|.$$

Proof. By definition of orthogonal projection, we have that

$$v - \operatorname{proj}_u v + \operatorname{proj}_u v = v,$$

where $v - \operatorname{proj}_u v$ and $\operatorname{proj}_u v$ are perpendicular. Hence we have that

$$\|v\|^2 = \|(v - \operatorname{proj}_u v) + \operatorname{proj}_u v\|^2 = \|v - \operatorname{proj}_u v\|^2 + \|\operatorname{proj}_u v\|^2 \geqslant \|\operatorname{proj}_u v\|^2.$$

As $\operatorname{proj}_u v = \frac{\langle u, v \rangle}{\|u\|^2} u$, simple rearrangements give

$$\|v\|^2 \geqslant \frac{|\langle v, u \rangle|^2}{\|u\|^2} \Rightarrow \|u\| \, \|v\| \geqslant |\langle v, u \rangle|^2.$$

$\square$

Remark. From the above propositions, $\|u\| = \sqrt{\langle u, u \rangle}$ is indeed a norm in the inner product space.

We will next discuss a bit into the projection onto a subspace as "best approximation".

Theorem. Assume $V = W \oplus W^\perp$. Then for all $v \in V$,

$$\|v - \mathrm{proj}_W v\| \leqslant \|v - w\|$$

for every $w \in W$, with equality obtained if and only if $w = \mathrm{proj}_W v$.

If we want to approximate $v \in V$ by some $w \in W$, the closest distance we can get is obtained by taking $\mathrm{proj}_W v$. In other words, $\mathrm{proj}_W v$ is the best approximation of v in W.

Proof. Consider the norm $\|v - w\|$. On a similar note to the proof of Cauchy-Schwarz, take

$$\|v - w\|^2 = \|(v - \mathrm{proj}_W v) + (\mathrm{proj}_W v - w)\|^2 .$$

Note that $v - \mathrm{proj}_W v \in W^\perp$ and $\mathrm{proj}_W v - w \in W$, so we can then apply Pythagorean theorem to obtain

$$\|v - w\|^2 = \|v - \mathrm{proj}_W v\|^2 + \underbrace{\|\mathrm{proj}_W v - w\|^2}_{\geqslant 0},$$

which proves the statement. The equality is taken if the second-term norm is zero. $\square$

Remark. The converse of the statement is also true (will be proven as a homework exercise.)

Corollary.

$$\|\mathrm{proj}_W v\| \leqslant \|v\| .$$

As $V = W \oplus W^\perp$ might not be true all the time, we may be interested to pose a question. A partial answer would be that W is a complete subspace of V. In particular, W finite-dimensional implies $V = W \oplus W^\perp$. (if W is finite-dimensional, the coordinate map provides an isomorphism between W with $\mathbb{R}^n$ or $\mathbb{C}^n$ while preserving the norms and distances. $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete.) Additionally, if V is complete but W is not, then $V \neq W \oplus W^\perp$.

Proposition. (*Gram-Schmidt*) If W is finite dimensional and $(w_1, \cdots, w_n)$ is an orthogonal basis of W,

$$\mathrm{proj}_W v = \sum_{j=1}^n \mathrm{proj}_{W_j} v.$$

Proof. It suffices to show that

$$v - \sum_{j=1}^n \mathrm{proj}_{W_j} v \in W^\perp.$$

Note that if a vector v^* is orthogonal to each of the basis elements of W, v^* is also orthogonal to W. To this end, consider each of the inner product

$$\left\langle v - \sum_{j=1}^n \mathrm{proj}_{W_j} v, w_k \right\rangle = -\sum_{j \in [n]-k} \left\langle \mathrm{proj}_{W_j} v, w_k \right\rangle + \left\langle v - \mathrm{proj}_{W_k} v, w_k \right\rangle .$$

The first term has $\left\langle \mathrm{proj}_{W_j} v, w_k \right\rangle = 0$ for all $j \in [n] - k$ as the projection $\mathrm{proj}_{W_j} v$ is parallel to w_j, which is perpendicular to w_k by the definition of orthogonal basis. Additionally, the second term is zero by definition of orthogonal projection. This proves the claim. $\square$

Beginning of March 10, 2023

In the last lecture, we discussed that if W is a complete subspace of a real or complex inner product space $(V, \langle\cdot,\cdot\rangle)$, then $V = W \oplus W^\perp$. In particular, if W is finite-dimensional with orthogonal basis $(w_1, \cdots, w_n)$, then

$$\mathrm{proj}_W v = \sum_{j=1}^n \mathrm{proj}_{W_j} v$$

for every $v \in V$. We will now consider the case in a general setting of infinite-dimensional inner vector spaces with a Schauder basis $(w_i)_{i=1}^\infty$. Recall that $(u_i)_{i=1}^\infty$ is a Schauder basis for a normed vector space $(V, \|\cdot\|)$ if for every $v \in V$ there exists a sequence $(c_i)_{i=1}^\infty$ of scalars such that

$$\lim_{n\to\infty} \left\| v - \sum_{i=1}^n c_i u_i \right\| = 0.$$

Theorem. Assume W has an orthogonal Schauder basis $(w_i)_{i=1}^\infty$. Put $W_n = \mathrm{span}(w_1, \cdots, w_n)$ for every $n \in \mathbb{N}$. Then for every $v \in V$,

$$\mathrm{proj}_W v = \sum_{i=1}^\infty \mathrm{proj}_{W_i} v = \lim_{n\to\infty} \sum_{i=1}^n \mathrm{proj}_{W_i} v = \lim_{n\to\infty} \mathrm{proj}_{W_n} v.$$

Proof. Denote $w = \mathrm{proj}_W v$. There exists $(c_i)_{i=1}^\infty$ such that $w = \lim_{n\to\infty} \sum_{i=1}^n c_i w_i$ with $\|w - \sum_{i=1}^n c_i w_i\| \overset{n\to\infty}{\to} 0$. Using the best approximation quantity, we have that

$$\left\| w - \sum_{i=1}^n \mathrm{proj}_{W_i} w \right\| \leqslant \left\| w - \sum_{i=1}^n c_i w_i \right\| \overset{n\to\infty}{\to} 0.$$

At this point, we know that $w = \mathrm{proj}_W v = \lim_{n\to\infty} \mathrm{proj}_{W_n} w$. Now we want to show that replacing w with v does *not* change the result. Here the key is to realize that $W_n \subset W$, so

$$\mathrm{proj}_{W_n} v = \mathrm{proj}_{W_n}(\mathrm{proj}_W v) = \mathrm{proj}_W(\mathrm{proj}_{W_N} v),$$

(to be proven as a homework exercise.) As

$$\mathrm{proj}_{W_n} v = \mathrm{proj}_{W_n} w \overset{n\to\infty}{\to} w = \mathrm{proj}_W v,$$

with the first equality obtained by the above property and the last equality directly from the definition. This proves the claim. □

Next we will discuss Bessel's inequality and Parseval's identity. Consider the following setup: $(V, \langle\cdot,\cdot\rangle)$ is a real or complex inner product space, and let $(e_n)_n$ be an orthonormal sequence in V with $W_n = \mathrm{span}(e_1, \cdots, e_n)$. Define

$$P_n(\cdot) := \mathrm{proj} W_n(\cdot), \quad f_i := \langle f, e_i \rangle.$$

Then

$$P_n f = \sum_{i=1}^n f_i e_i.$$

Additionally, note that

$$\|P_n f\|^2 = \left\langle \sum_{i=1}^n f_i e_i, \sum_{i=1}^n f_i e_i \right\rangle = \sum_{i=1}^n \sum_{j=1}^n f_i \overline{f_j} \langle e_i, e_j \rangle = \sum_{i=1}^n |f_i|^2.$$

With the above clarifying notations, we can introduce Bessel's inequality.

Theorem. (*Bessel*)

$$\lim_{n\to\infty} \|P_n f\|^2 = \sum_{i=1}^{\infty} |f_i|^2 \leqslant \|f\|^2 .$$

Lemma. (*Riemann-Lebesgue*) $f_i \to 0$ as $i \to \infty$.

Theorem. (*Parseval*) If $P_n f \to f$ and $P_n g \to g$ as $n \to \infty$, then

$$\langle f, g\rangle = \sum_{i=1}^{\infty} f_i \overline{g_i} = \langle (f_i)_i, (g_i)_i \rangle .$$

Note that Riemann-Lebesgue lemma is an immediate consequence of Bessel's inequality by series convergence.

Remark. The equality for Bessel's lemma is obtained where $P_n f \to f$.

Proof of Bessel's Inequality. Bessel's lemma is (sort of) already been proven, as $\|P_n f\| \leqslant \|f\|$ for every $n \in \mathbb{N}$ by the definition of orthogonal projection. Regarding the equality, take

$$\|f\|^2 = \|f - P_n f + P_n f\|^2 = \|f - P_n f\|^2 + \|P_n f\|^2 .$$

The equality can only be taken when $\|f - P_n f\|^2 \to 0$, i.e., $P_n f \to f$ as $n \to \infty$. □

Proof of Parseval's Identity. Here we apply the polarization identity, which tells us that every inner product can be written as some combination of norms. With $P_n f \to f$ and $P_n g \to g$,

$$\|f+g\|^2 = \sum_{n=1}^{\infty} |f_n + g_n|^2 = \|(f_n + g_n)_n\|_{\ell^2(\mathbb{N};\mathbb{C})} \Rightarrow \langle f, g\rangle = \frac{1}{4}\left[\|f+g\| + \cdots\right] = \frac{1}{4}\left[\|(f_i+g_i)_i\|_{\ell^2(\mathbb{N};\mathbb{C})} + \cdots\right] = \langle (f_i)_i, (g_i)_i\rangle .$$

□

Now we discuss the statement in a different setup. Consider $(V, \langle\cdot,\cdot\rangle)$ a Hilbert space, and $(e_i)_i$ an orthonormal Schauder basis. Then $P_n f \to f$ in $(V, \langle\cdot,\cdot\rangle)$ for every $f \in V$. Consequently,

$$\sum_{i=1}^{\infty} |f_i|^2 = \|f\|^2 ; \ \ \langle f, g\rangle = \sum_{i=1}^{\infty} f_i \overline{g_i}.$$

The map $L : (V, \langle\cdot,\cdot\rangle)$ to $(\ell^2(\mathbb{N}; \mathbb{C}, \langle\cdot,\cdot\rangle)$ determined by

$$L(f) = (f_i)_i$$

is an isometry of Hilbert spaces.

Beginning of March 20, 2023

This lecture will be a crash course on Fourier series. The goal is to represent $f : [-\pi, \pi] \to \mathbb{C}$ in L^2 by

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$

Remark. We cannot directly use the Stone-Weierstraß theorem. Not to say the limitations (f may *not* be periodic or continuous), Namely, we want to look for a <u>single</u> choice of two-sided sequence of coefficients $(c_n)_{n=-\infty}^{\infty}$ regardless of the error tolerance.

One way to think about Fourier series is that we consider the low-frequency truncations

$$S_N f(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}$$

while cutting off the high-frequency terms. This can be done by truncating the series of exponentials from $\mathbb{Z}$.

The setting used in Fourier approximation is $f \in (L^2_{\mathcal{R}}([-\pi,\pi];\mathbb{C},\langle\cdot,\cdot\rangle)$.
Additionally, $f \sim g$ if $f = g$ a.e., $f, g \in \mathcal{R}[-\pi,\pi]$. The inner product is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\,\mathrm{d}\theta = \frac{1}{2\pi}\left\|f\overline{g}\right\|_{L^1,[-\pi,\pi]}.$$

New notations include

$$e_n(\theta) = e^{in\theta},\quad \langle e_n, e_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(n-m)\theta}\,\mathrm{d}\theta = \begin{cases}1, & n = m\\ 0, & n \neq m.\end{cases}$$

Note that $(e_n)_{n=-\infty}^{\infty}$ is orthonormal and linearly independent.

Definition. The **Fourier coefficients** are defined as

$$\hat{f}(n) = \langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,\mathrm{d}\theta.$$

Note that the projection of f on e_n takes

$$\mathrm{proj}_{e_n} f = \frac{\langle f, e_n\rangle}{\|e_n\|^2}e_n = \hat{f}(n)e_n.$$

As a result, the N-th partial sum of f, denoted $S_n f$, is

$$S_N f = \mathrm{proj}_{W_N}(f) = \sum_{n=-N}^{N}\hat{f}(n)e_n,\quad w_N = \mathrm{span}(e_n)_{n=-N}^{N}.$$

Even if f is real-valued,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,\mathrm{d}\theta$$

may *not* be real-valued. However, we **claim** that the N-th partial sum of f is real-valued. Indeed, consider

$$\overline{\hat{f}(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{in\theta}\,\mathrm{d}\theta = \hat{f}(-n).$$

Thus $\overline{\hat{f}(n)e_n} = \hat{f}(-n)e_{-n}$. Each time we extend N by one, we add a term in both directions, which can be summed up as such:

$$S_N f = \hat{f}(0)e_0 + \sum_{n=1}^{N}\hat{f}(n)e_n + \hat{f}(-n)e_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta + \sum_{n=1}^{N} 2\mathrm{Re}(\hat{f}(n)e_n).$$

Suppose we already know that

$$p(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta} = \mathrm{proj}_{W_N}(p),$$

we can immediately see that $\hat{p}(m) = c_m$ for $|m| \leqslant N$, and $\hat{p}(m) = 0$ for $|m| > N$. Additionally $S_M p = p$ for $M \geqslant N$.

Theorem. If $f \in L^2_{\mathcal{R}}$, then

$$\lim_{N\to 0}\|S_n f - f\|_{L^2} = 0.$$

Proof. Let $g \in C_{\text{per}}([-\pi, \pi])$, then consider the sum

$$f - S_N f = f - g + g - p + p - S_N(p) + S_N(p) - S_N(g) + S_N(g) - S_N(f).$$

and our goal is to (1) make $f - g$ small in L^2, and (2) control $g - p$ in $\|\cdot\|_u$ with Stone-Weierstraß. Choosing N large makes $p - S_n(p)$ zero, then

Blue part. Here we choose $g \in C_{\text{per}}([-\pi, \pi])$ such that $\|f - g\|_{L^2} < \frac{\varepsilon}{4}$. (This is done by homework exercise 3.5.1.)

Problem: Leslie 3.5.1. Suppose $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ is T-periodic, and assume $1 \leqslant p < \infty$. Show that given $\varepsilon > 0$ there exists a continuous T-periodic function g such that $\|g\|_u \leqslant 4\|f\|_u$ and

$$\int_0^T |f(x) - g(x)|^p \, \mathrm{d}x < \varepsilon.$$

Red part. Here we consider $p \in P_{\text{trig}}([-\pi, \pi])$ such that $\|g - p\|_u < \frac{\varepsilon}{4}$. As the uniform norm is stronger than that L^2 norm, such p works as we apply Stone-Weierstraß.

Purple part. Here p is a polynomial; hence by the previous observation we can simply take $N \geqslant \deg(p)$ to make the difference zero.

Orange part. Note that $\|S_n(p) - S_n(g)\| = \|S_n(p - g)\|$. As $S_n(\cdot)$ is a projection operator of $(\cdot)$ onto W_N, it cannot increase length, so $\|S_n(p - g)\| \leqslant \|p - g\| < \frac{\varepsilon}{4}$.

Green part. Again, $\|S_n(g) - S_n(f)\| = \|S_n(g - f)\| \leqslant \|g - f\| < \frac{\varepsilon}{4}$ if we consider the L^2 norm.

Summing all five parts, we have

$$\|f - S_N f\| \leqslant \|f - g\| + \|g - p\| + \|p - S_N(p)\| + \|S_N(p) - S_N(g)\| + \|S_N(g) - S_N(f)\| < \varepsilon,$$

as needed. $\square$

The above is all true; however, as soon as we want to write

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n,$$

the statement become problematic - the right-hand side may *not* be a Riemann-integrable function. In fact, f may differ from the infinite series by a set of measure zero, which causes problems when we try to perform integration.

We will end the lecture with three propositions without proof.

Proposition. Fourier coefficients are uniquely determined, i.e., if

$$\lim_{N \to \infty} \left\| \sum_{n=-N}^{N} c_n e_n - f \right\| = 0,$$

then $c_n = \hat{f}(n)$ for every n.

Corollary. If $f, g \in \mathcal{R}([-\pi, \pi])$ and $\hat{f}(n) = \hat{g}(n)$ for every n, then $f = g$ a.e..

Proposition. (*Parseval*) If $f, g \in L^2_{\mathcal{R}}$, then

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\,\mathrm{d}\theta = \left\langle (\hat{f}(n))_{n=-\infty}^{\infty}, (\hat{g}(n))_{n=-\infty}^{\infty}\right\rangle_{\ell^2(\mathbb{Z};\mathbb{C})} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

Specifically,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(\theta)\right|^2\,\mathrm{d}\theta = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2.$$

Beginning of March 22, 2023

Consider two F-vector spaces X and Y. We define

$$\mathcal{L}(X, Y) = \{\text{all linear transformations from } X \text{ to } Y\}.$$

The primary question here is: when is $T \in \mathcal{L}(X, Y)$ continuous? The answer is "not always", unless X is finite-dimensional. We will primarily discuss infinite-dimensional vector spaces.

Example. Consider the vector spaces with norms

$$(X_0 : \|\cdot\|_{X_0}) = (C^1[0,1], \|\cdot\|_u), (X_1 : \|\cdot\|_{X_1}) = (C^1[0,1], \|\cdot\|_{C^1}), (Y, \|\cdot\|_Y) = (C[0,1], \|\cdot\|_u),$$

and the transformation $T_i : X_i \to Y$ defined by $T_i(f) = f'$ for $i = 0, 1$. We **claim** that T_1 is continuous but T_0 is not. Particularly,

$$\|T_1 f - T_1 g\|_Y = \|f' - g'\|_u \leqslant \|f - g\|_{X_1},$$

and the proof follows from an ε-δ argument. However, if we try to run

$$\|f' - g'\|_u \overset{?}{\leqslant} \|f - g\|_u,$$

this is indeed untrue. Indeed, taking some function with low uniform norm but high derivative, such as $f_n = \frac{1}{n}\sin nx$, gives us an explicit counterexample. Particularly $f_n \to 0$ but $T_0 F_n \nrightarrow 0$.

Theorem. Let $T : X \to Y$ be a linear transformation between the normed F-vector spaces. The following statements are equivalent:

(1) T is *bounded* that there exists $C > 0$ such that $\|Tx\|_Y \leqslant C\,\|x\|_X$ for every $x \in X$.

(2) T is continuous.

(3) T is continuous at zero.

Proof: (1) implies (2). Assume that T is bounded. Take $x_1, x_2 \in X$, we wish $Tx_1 - Tx_2$ small in Y whenever $x_1 - x_2$ is small in X. Applying linearity, we have that

$$\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leqslant C\,\|x_1 - x_2\|_X.$$

Take $\|x_1 - x_2\|_X < C^{-1}\varepsilon$ for every $\varepsilon > 0$ suffices.

Proof: (3) implies (1). Assume T is continuous at zero. Here we pick $\varepsilon = 1$. Choose $\delta > 0$ such that $\|x\|_X < \delta \Rightarrow \|Tx\|_Y \leqslant 1$. For any $x \in X - \{0\}$, consider $\tilde{x} \in \partial B(0, \delta)$ defined by

$$\tilde{x} = \frac{\delta x}{\|x\|}.$$

Thus considering the linear transformation $T\tilde{x}$, we have that

$$\left\| T \frac{\delta x}{\|x\|_X} \right\|_Y \leqslant 1 \Rightarrow \|Tx\|_Y \leqslant \delta^{-1} \|x\|_X .$$

Hence taking $C = \delta^{-1}$ suffices. □

Definition. Define

$$\mathcal{B}(X, Y) = \{\text{all bounded linear transformations from } (X, \|\cdot\|_X) \text{ to } (Y, \|\cdot\|_Y)\} .$$

The **operator norm** of $T \in \mathcal{B}(X, Y)$ is defined by

$$\|T\|_{X \to Y} := \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

Proposition.

$$\sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X \leqslant 1} \|Tx\|_Y = \inf \{C \geqslant 0 : \|Tx\|_Y \leqslant C \|x\|_X\} .$$

The proof will be left as an exercise. We consider some of its implications in the lecture.

Corollary. If $T : X \to Y$ is bounded, then $\|Tx\|_Y \leqslant \|T\| \|x\|_X$.

Corollary. If $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$, then

$$ST \in \mathcal{B}(X, Z)$$

with operator norm

$$\|ST\|_{X \to Z} \leqslant \|S\|_{Y \to Z} \|T\|_{X \to Y} .$$

Remark. ST is a composition of transformation. STx means "transforming x to Tx, then Tx to STx.

Proposition. Suppose $T \in \mathcal{L}(X, Y)$. If X is finite-dimensional, then $T \in \mathcal{B}(X, Y)$.

Proof. Consider a basis for X: $(x_1, \cdots, x_n)$. Then for every $x \in X$,

$$\|Tx\|_Y = \|T(c_1 x_1 + \cdots + c_n x_n)\|_Y \leqslant \sum_{j=1}^{n} |c_j| \|Tx_j\|_Y \leqslant \max_{j \in [n]} |c_j| \left(\sum_{j=1}^{n} \|T_{x_j}\|_Y \right).$$

The first term $\max_{j \in [n]} |c_j|$ is a norm that is equivalent to $\|\cdot\|_X$. Additionally, $\sum_{j=1}^{n} \|T_{x_j}\|_Y$ is a fixed value, so $\|Tx\|_Y$ is indeed bounded above; hence $T \in \mathcal{B}(X, Y)$. □

Proposition. Let X and Y be normed vector spaces. If $(Y, \|\cdot\|_Y)$ is complete, then so is $(\mathcal{B}(X,Y), \|\cdot\|_{X\to Y})$.

Beginning of March 24, 2023

Today we will cover isomorphisms and matrices as transformation on finite-dimensional spaces.

Definition. An **isomorphism** is a bijection between two objects that preserves structure. Specifically,

- A **topological isomorphism** (or a **homeomorphism**) preserves topology, where f and f^{-1} map open sets to open sets; this is equivalent to that f and f^{-1} are continuous.
- A **vector space isomorphism** preserves vector space structure. Linear bijections always do this.
- A **normed vector space isomorphism** preserves both topology and vector space structure.

Remark. Technically the above definition on normed vector space isomorphism is *not* the "right" one. See below.

Definition. Let $T : X \to Y$ is a linear transformation between two normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$. If T is surjective and there exists $c, C > 0$ such that

$$c\|x\|_X \leqslant \|Tx\|_Y \leqslant C\|x\|_X$$

for every $x \in X$, then we say that T is an **isomorphism** on normed vector spaces.

Proposition. $T : X \to Y$ is a normed vector space isomorphism if and only if it is a vector space isomorphism and a homeomorphism.

Proof. We only prove ($\Rightarrow$) here. ($\Rightarrow$) To show injectivity, consider, for $x_1 \neq x_2$,

$$\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \geqslant c\underbrace{\|x_1 - x_2\|_X}_{>0} > 0.$$

As T is a linear bijection, it is a vector space isomorphism.
To show topological isomorphism, we have that $\|Tx\|_Y \leqslant C\|x\|_X$ which implies T continuous. Considering the continuity of T^{-1}, we have that

$$\left\|T^{-1}y\right\|_Y = \frac{1}{c}\left(c\left\|T^{-1}y\right\|_X\right) \leqslant \frac{1}{c}\left\|T(T^{-1}y)\right\|_Y = \frac{1}{c}\|y\|_Y.$$

Therefore T^{-1} also has finite operator norm, hence it is continuous. ☐

We denote the set of all normed vector space isomorphisms $T : X \to Y$ as

$$\Omega(X,Y) = \{\text{Normed Vector Space Isomorphisms} T : X \to Y\}.$$

$\Omega(X,Y) \subset \mathcal{B}(X,Y)$.

Theorem. Assume $T \in \Omega(X,Y)$, $S \in \mathcal{B}(X,Y)$, and

$$\|S - T\|_{X \to Y} < \|T^{-1}\|_{Y \to X}^{-1}.$$

Then $S \in \Omega(X,Y)$ and

$$S^{-1} = \sum_{n=0}^{\infty} (\mathrm{id}_X - T^{-1}S)^n T^{-1}.$$

Corollary. $\Omega(X,Y)$ is an open subset of $\mathcal{B}(X,Y)$.

Proof. Take $r = \|T^{-1}\|_{Y \to X}^{-1}$ for each $T \in \Omega(X,Y)$. □

Corollary. $\Omega(X,Y)$ is homeomorphic to $\Omega(Y,X)$.

Proof. Consider the continuous map $T \to T^{-1}$. □

We then work with matrices: a quick linear algebra review. Formally, matrices are linear transformations on finite-dimensional vector spaces. Consider the following setting.

$$F^{m \times n} = \{m \times n \text{ matrices with entties in } F\}.$$

A matrix $A \in F^{m \times n}$ has m rows and n columns, with

$$A = (a_{i,j}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

The algebraic operations of matrices is omitted here. Particularly, if $A \in F^{m \times n}$ and $\mathbf{c} = [c_1, \cdots, c_n]^T$,

$$A\mathbf{c} = \sum_{j=1}^{n} c_j \mathrm{col}_j(A) \in F^m.$$

We will also go over change of basis. Let X be a finitely-dimensional vector space, and let $U = (u_1, \cdots, u_n)$ and $U' = (u_1', \cdots, u_n')$ be bases. The function $\varphi_U : X \to F^n$ take

$$[u]_U = \varphi_U(c_1 u_1 + \cdots + c_n u_n) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Then considering $[u]_{U'}$, we have that

$$[u]_{U'} = [c_1 u_1 + \cdots + c_n u_n]_{U'} = \sum_{j=1}^{n} c_j [u_j]_{U'}.$$

Here we define the change-of-basis matrix $P_{U' \leftarrow U}$ by

$$\mathrm{col}_j(P_{U' \leftarrow U}) = [u_j]_{U'}.$$

Then for this change-of-basis transformation, we have that

$$[u]_{U'} = P_{U' \leftarrow U}[u]_U.$$

To this end, let $T : X \to Y$ be a normed vector space; let X and Y be finite-dimensional vector spaces with basis $U = (u_1, \cdots, u_n)$ and $V = (v_1, \cdots, v_n)$, respectively. Additionally, let

$$[u]_U = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

With the transformation $T : X \to Y$,

$$[Tu]_V = \sum_{j=1}^{n} c_j [Tu_j]_V = A[u]_U,$$

where A is the matrix defined by

$$\mathrm{col}_j(A) = [Tu_j]_V.$$

Formally, A is the **matrix representation** of T with respect to U and V. Further, the matrix representation of T with respect to U' and V' can be obtained by combining the two conclusions above:

$$[Tu]_{V'} = P_{V' \leftarrow V}[Tu]_V = P_{V' \leftarrow V} A[u]_U = P_{V' \leftarrow V} A P_{U \leftarrow U'}[u]_{U'}.$$

We present an important conclusion regarding transformations: If X, Y are finite-dimensional F-vector spaces, then

$$\mathcal{L}(X, Y) \cong F^{m \times n}.$$

In particular, consider the transformation $T \mapsto A$ that maps T to the matrix of transformation A. Its inverse is

$$\varphi_V^{-1} \circ L_A \circ \varphi_U \leftarrow\!\shortmid A.$$

Lastly, we give some norms on $F^{m \times n}$: for

$$A = (a_{i,j}) \in F^{m \times n},$$

- The operator norm on $\mathcal{L}(F^n, F^m)$ takes $\|A\| = \|L_A\|_{F^n \to F^m}$;
- The ℓ^1 norm, or $\|\cdot\|_1$, is defined as
$$\|A\|_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{ij}\|;$$
- The Frobenius (ℓ^2) norm is defined as
$$\|A\|_2^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{ij}\|^2.$$

  The Frobenius norm also preserves the inner product structure from F^{nm}:
$$\langle A, B \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \overline{b_{ij}}.$$

Proposition. The Frobenius norms satisfy

$$\|A^T B\|_2 \leqslant \|A\|_2 \|B\|_2.$$

Beginning of March 27, 2023

**Prof. Leslie was not available today; instead the class is taught by Prof. Joshua Swanson.*

Today we will start with multivariable differentiation. Recall the classical definition of derivative in 425a. The derivative of $f:(a,b)\to\mathbb{R}$ at $x_0\in(a,b)$, if it exists, is defined as

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

The question we have here in 425b is: what about $f:U\to Y$ where U is an open subset of normed vector space X? First noting if we change $\mathbb{R}$ in the classical definition to Y, the statement would *not* change much. However, there are some complications when replacing the domain by a normed vector space. In fact, we cannot directly "add h" to some x_0; the idea is to consider x_0+hz in the direction of some fixed $z\in X$.

Definition. The **Gâteaux derivative** of $f:U\to Y$ in the direction of $z\in X$ at $x_0\in U$ is

$$D_zf(x_0)=\lim_{h\to 0}\frac{f(x_0+hz)-f(x_0)}{h},$$

provided that the derivative exists in $(Y,\|\cdot\|_Y)$.

Remark. The Gâteaux derivative is the generalization of directional derivative from Calculus III.

The Gâteaux derivative is, in fact, a special case of the classic derivative. Given $f:U\to Y$, $x_0\in U$ and $z\in X$, we can define the auxiliary function

$$f_z:(-\varepsilon,\varepsilon)\to Y \text{ by } f(h)=F(x_0+hz).$$

Then, if the limit exists,

$$D_zf(x_0)=\lim_{h\to 0}\frac{f(x_0+hz)-f(x_0)}{h}=\lim_{h\to 0}\frac{f_z(h)-f_z(0)}{h}=f_z'(0).$$

Now we discuss the issue of the Gâteaux derivative. Namely, even if $D_zf(x_0)$ exists for all $z\in X$, f might not be continuous at x_0.

Example. Define the piecewise function $f:\mathbb{R}^2\to\mathbb{R}^2$ as

$$f(\mathbf{x})=\begin{cases}\frac{x_1^2x_2}{x_1^4+x_2^2} & \text{if } \mathbf{x}\neq\mathbf{0};\\ 0 & \text{if } \mathbf{x}=\mathbf{0}.\end{cases}$$

Then

$$\frac{f(\mathbf{0}+h\mathbf{z})-f(\mathbf{0})}{h}=\frac{(hz_1)^2(hz_2)}{h((hz_1)^4+(hz_2)^2)}=\frac{z_1^2z_2}{h^2z_1^2+z_2^2}\overset{h\to 0}{\to}\frac{z_1^2}{z_2},$$

hence $D_zf(\mathbf{0})=z_1^2z_2^{-1}$ if $z_2\neq 0$ and $D_zf(\mathbf{0})=0$ if $z_2=0$. But f is not continuous at $\mathbf{0}$. (Why?)

Remark. The Gâteaux derivative is not sufficient for nice Taylor expansions if we want to find the "best linear approximation" to $f:U\to Y$ near $x_0\in U$.

Definition. Let X, Y be real normed vector spaces, $U \subset X$ open, and $g : U \to Y$ a function. We say that g is **Fréchet differentiable** at $x_0 \in U$ with derivative $T \in \mathcal{B}(X, Y)$ if

$$\lim_{z \to 0} \frac{\|g(x_0 + z) - g(x_0) - Tz\|_Y}{\|z\|_X} = 0.$$

That is, as the norm of z goes to zero, we are essentially reuiring the error of best approximation Tz to tend to zero. Such T is unique, so we denote it by $g'(x_0)$.

Remark. We can take any path with $\|z\| \to 0$, whereas the Gâteaux derivative only uses "straight lines" (as seen in its decomposition into auxiliary functions.) Hence the Fréchet derivative is more global in some sense.

Remark. We can also define continuous differentiability here: $g \in C^1(U; Y)$ means $g' : U \to \mathcal{B}(X, Y)$ is continuous.

Proposition. Assume $g : U \to Y$ is differentiable at $x_0 \in U \subset X$. Then:

(a) g is continuous at x_0;

(b) $g'(x_0)z = D_z g(x_0)$ for every $z \in X$.

Proof of (a). Consider

$$\|g(x_0 + z) - g(x_0)\|_Y \leqslant \|g(x_0 + z) - g(x_0) - g'(x_0)z\|_Y + \|g'(x_0)z\|_Y.$$

The second term is bounded by $\|g'(x_0)\|_{X \to Y} \|z\|_X$, which $\to 0$ with the operator norm $< \infty$ and $\|z\|_X \to 0$. On the other hand, considering the first term we have

$$\|g(x_0 + z) - g(x_0) - g'(x_0)z\|_Y \leqslant \|z\|_X \frac{\|g(x_0 + z) - g(x_0) - g'(x_0)z\|}{\|z\|_X},$$

with the right-hand side $\to 0$ as $z \to 0$.

Proof of (b). We check if $g'(x_0)z$ satisfies the requirement of $D_z g(x_0)$:

$$\left\| \frac{g(x_0 + hz) - g(x_0)}{h} - g'(x_0)z \right\|_Y = \frac{\|g(x_0 + hz) - g(x_0) - g'(x_0)(hz)\|_Y}{\|hz\|_X} \|z\|_X,$$

with the first term $\to 0$ as $h \to 0$ by the definition of the Fréchet derivative. Therefore the Fréchet derivative agrees with the Gâteaux derivative, as desired. □

Beginning of March 29, 2023

Recall that a function $g : U \to Y$ is Fréchet differentiable at $x_0 \in U$ with derivative $T \in \mathcal{B}(X, Y)$ if

$$\lim_{z \to 0} \frac{\|G(x_0 + z) - G(x_0) - Tz\|_Y}{\|z\|_X} = 0.$$

Such derivative is unique. Near x_0, we have that

$$g(x) + g'(x_0)(x - x_0) + o(\|x - x_0\|_X) \text{ as } x \to x_0 \in X.$$

Definition. $f(z) = o(g(z))$ as $z \to a$ means

$$\lim_{z \to a} \frac{f(z)}{g(z)} = 0.$$

Example. $x^2 = o(x)$ as $x \to 0$.

Additionally, if we can write

$$g(x) = g(x_0) + T(x - x_0) + o(\|x - x_0\|_X)$$

and $T \in \mathcal{B}(X, Y)$, then $T = g'(x_0)$.

Example. Consider $g : \mathbb{R}^{m\times n} \to \mathbb{R}^{n\times n}$ defined by $g(A) = A^T A - I$. Is g differentiable? What is $g'(A)$?

Sol. We consider the difference $d = g(A + H) - G(A)$, which is

$$d = ((A + H)^T (A + H) - I) - (A^T A - I) = H^T A + A^T H + H^T H.$$

We suspect that $H^T H = o(\|H\|_2)$. To reach the conclusion, we guess $TH = H^T A + A^T H$ and $g'(A) = T$. To validate the condition, we need to show that $T \in \mathcal{B}(\mathbb{R}^{m\times n}, \mathbb{R}^{n\times n})$ and $g(A + H) = g(A) + TH + o(\|H\|_2)$. The first statement is almost trivial as $\mathbb{R}^{m\times n}$ is finite-dimensional. For the second statement, we need to check that

$$\frac{\left\|H^T H\right\|_2}{\|H\|_2} \overset{H\to 0}{\to} 0.$$

Here we use the fact that $\left\|H^T H\right\|_2 \leqslant \|H\|_2^2$, which gives $\|H\|_2 \to 0$ as $H \to 0$, as desired.

Example. Consider $g : C([a, b], \|\cdot\|_u) \to C^1([a, b], \|\cdot\|_{C^1})$ defined by $g(f)(x) = \int_a^x f(t)^2 \, \mathrm{d}t$. Is g differentiable? What is $g'(f)$?

Sol. We consider the difference $d = (g(f + h) - g(f))(x)$. This equates

$$d = \int_a^x (f + h)(t)^2 - f(t)^2 \, \mathrm{d}t = \int_a^x 2f(t)h(t) + h(t)^2 \, \mathrm{d}t.$$

We guess that $g'(f) = 2\int_a^x f(t)h(t) \, \mathrm{d}t$. To this end, note that

$$\|Th(x)\|_{C^1} = \left\| \int_a^x 2f(t)h(t) \, \mathrm{d}t \right\|_u + \|2fh\|_u \leqslant 2(x - a) \|f\|_u \|h\|_u + 2 \|f\|_u \|h\|_u \leqslant 2\,(b - a + 1) \|f\|_u \|h\|_u .$$

Therefore $T = g'(f)$ is both a well-defined and a continuous linear transformation. Again, we need to check that

$$\frac{\left\|\int_a^x h(t)^2 \, \mathrm{d}t\right\|_{C^1}}{\|h\|_u} \overset{h\to 0}{\to} 0.$$

The C^1 norm of the numerator is bounded above by $(b - a + 1) \|h\|_u^2$, which gives $c\|h\|_u \to 0$ as $h \to 0$.

Next we will discuss a bit about chain rule. Consider U, V that are subsets of X and Y, respectively. Let $g : U \to Y$ and $h : V \to Z$. Additionally, $x_0 \in U$ and $g(x_0) \in V$; g differentiable at x_0 and h differentiable at $g(x_0)$. Then,

Theorem. (*Chain rule*)

$$(h \circ g)'(x_0) = h'(g(x_0)) \circ g'(x_0).$$

Proof. As g is differentiable at x_0, we know that

$$g(x_0 + a) = g(x_0) + g'(x_0)a + \underbrace{\varepsilon_g(a)}_{o(\|a\|_X)}.$$

Similarly, as h is differentiable at $g(x_0)$,

$$h(g(x_0) + b) = h(g(x_0)) + h'(g(x_0))b + \underbrace{\varepsilon_h(b)}_{o(\|b\|_y)}.$$

Therefore, considering the composition $h \circ g$, we naturally have that

$$h(g(x_0 + a)) - h(g(x_0)) = h(g(x_0) + g'(x_0)a + \varepsilon_g(a)) - h(g(x_0)).$$

Now that as $g'(x_0)a + \varepsilon_g(a) = b$, the above expression equates

$$h'(g(x_0))\left[g'(x_0)a + \varepsilon_g(a)\right] + \varepsilon_h(b).$$

It may take some work to prove that the infinitesimality of $\varepsilon_g(a)$ and $\varepsilon_h(b)$ are preserved under transformations, but the intuition should be clear; this leaves

$$(h \circ g)'(x_0) = h'(g(x_0)) \circ g'(x_0),$$

as desired. □

Beginning of April 3, 2023

Recall the chain rule between normed vector spaces:

Proposition. (*Chain rule*)

$$(h \circ g)'(x) = h'(g(x)) \circ g'(x).$$

Corollary. Let U be open in X and let V be open in Y. Function $g : U \to V$ is a bijection, differentiable at $x \in U$ and g^{-1} differentiable at $g(x) \in V$. Then $g'(x)$ is invertible with

$$g'(x)^{-1} = g^{-1}(g(x)).$$

In particular, $g'(x)$ is a normed vector space isomorphism.

Proof. Proof left as a homework exercise.

Proposition. Assume $g \in C^1(\mathbb{R})$, $f \in B(U,\mathbb{R})$ and $G : (B(U,\mathbb{R}), \|\cdot\|_u) \to (B(U,\mathbb{R}, \|\cdot\|_u)$, $G(f) = g \circ f$. Then G is differentiable at f for every $f \in B(U,\mathbb{R}, \|\cdot\|_u)$ and

$$[G'(f)z](x) = g'(f(x))z(x).$$

Remark. If $g \in C^1$, then for every $r > 0$,

$$\lim_{h\to 0} \sup_{|y|\leqslant r} \frac{|g(y+h) - g(y) - g'(y)h|}{|h|} = 0.$$

Proof. We want to show that

$$\lim_{z\to 0} \frac{\|G(f+z) - G(f) - (g' \circ f)\|_u}{\|z\|_u} = \frac{\sup_{x\in U} |g(f(x) + z(x)) - g(f(x)) - g'(f(x))z(x)|}{\sup_{x\in U} |z(x)|}.$$

If we restrict the attention to $|h| \leqslant \|z\|_u$, we have that

$$\lim_{z\to 0}(\cdot) \leqslant \sup_{|h|\leqslant\|z\|_u} \left(\sup_{|y|\leqslant\|f\|_u} \frac{|g(y+h) - g(y) - g'(y)h|}{|h|} \right).$$

□

The above proposition is certainly useful. Consider the following example.

Example. Consider $g : (C[a,b], \|\cdot\|_u) \to (C^1[a,b], \|\cdot\|_{C^1})$ defined by $[G(f)](x) = \int_a^x f(t)^2 \, \mathrm{d}t$. What is $G'(f)z$?

Sol. We let $G_1(f) = f^2$, and $G_2(\tilde{f})(x) = \int_a^x \tilde{f}(t) \, \mathrm{d}t$. Thus $G = G_2 \circ G_1$, so

$$G'(f)z = [G_2'(G_1(f)) \circ G_1'(f)]\, z.$$

First note that $G_1'(f)z = (g' \circ f)z = 2fz$. Then considering $G_2'(F_1(f))$, a linear transformation, we have that simply $G_2'(G_1(f)) = G_2$, so

$$G'(f)z = G_2(2fz) \Rightarrow (G'(f)z)(x) = \int_a^x 2f(t)z(t) \, \mathrm{d}t.$$

We now talk a bit about partial derivatives.

Definition. Let X and Y be finite-dimensional real vector spaces with basis $U = (u_1, \cdots, u_n)$ and $V = (v_1, \cdots, v_n)$. Let P be open in X; for a function $f : P \to Y$, we can define

$$f(x) = \sum_{i=1}^{m} f_i(x) v_i,$$

where $f_i(x)$ are the **components** of f with respect to that basis V. Naturally,

$$[f(x)]_V = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}, \quad f_i(x) = [f(x)]_V \cdot e_i.$$

Definition. The Gâteaux derivative of f can now be defined by

$$D_{u_j} f(x) = D_{u_j} \left(\sum_{i=1}^{m} f_i(x) v_i \right) = \sum_{i=1}^{m} D_{u_j} f_i(x) v_i,$$

and the i-th component of the derivative is

$$\left(D_{u_j} f(x) \right)_i = D_{u_j} f_i(x).$$

Here the expression $D_{u_j} f_i(x) = \partial_j f_i(x)$ is the **partial derivative** of f with respect to the basis U and V.

Definition. If we collect all mn partial derivatives $\partial_j f_i$ in an $m \times n$ matrix, we get the **Jacobian matrix**:

$$Jf(x) = \begin{bmatrix} \partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(x) & \cdots & \partial_n f_m(x) \end{bmatrix} \in \mathbb{R}^{m \times n}.$$

The Jacobian matrix is the matrix representation of $f'(x)$ with respect to bases U and V. Namely,

$$[f'(x)z]_V = Jf(x) \, [z]_U \, .$$

Beginning of April 5, 2023

A correction from last time's notes: if g is differentiable at x and g^{-1} is also differentiable at $g(x)$, then $g'(x)$ is a normed vector space isomorphism *regardless of dimension*.

Last time we were left off with the Jacobian matrix. Namely, in finite dimensions, $Jf(x)$ is the matrix representation of $f'(x)$ whenever f is differentiable at x. Particularly, we have the derivative formula:

$$[f'(x)]_V = Jf(x) \, [z]_U \Rightarrow f'(x)z = \varphi_V^{-1} \left(Jf(x) \, [z]_U \right),$$

where φ_V^{-1} is the inverse coordinate vector to V.

Proposition. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function, and U and V are basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then $Jf(x)$ is the matrix representation of $f'(x)$.

Proof. Suppose A is the matrix representation of $f'(x)$ with respect to basis U and V. Then

$$\mathrm{col}_j A = Ae_j = A \, [u_j]_U = [f'(x)u_j]_V = [\partial_j f(x)]_V \, .$$

The partial ∂_j of $f(x)$ takes the form of

$$[\partial_j f(x)]_V = \left[\partial_j \sum_{i=1}^{m} f_i(x) v_i \right]_V = \sum_{i=1}^{m} \partial_j f_i(x) \, \underbrace{[v_i]_V}_{e_i} \, .$$

Note that $\sum_{i=1}^{m} \partial_j f_i(x) e_i$ is simply $\mathrm{col}_j Jf(x)$, which completes the proof. □

Remark. Note the change in dimensions that are done through the matrix A. $[u_j]_U$ is a $\mathbb{R}^n$-column (hence U), and A is a $\mathbb{R}^{m \times n}$ matrix. Hence the product $A \, [u_j]_U$ returns a $\mathbb{R}^m$-column (hence V).

Until now, we have learned that if we know a function is differentiable at x_0, then we can compute the derivative using the Jacobian. However, the existence of $Jf(x)$ does not imply differentiability. This brings the question: if $Jf(x)$ exists and we have additional information, can we conclude f is differentiable at x?

Proposition. Let X and Y be finite dimensional real normed vector spaces with basis $U = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_m)$, respectively. $f : P \to Y$ is C^1 at $x_0 \in P \subset X$ if and only if $x \mapsto Jf(x)$ is continuous at x_0 (with respect to any norm) if and only if all partial derivatives are continuous at x_0.

Remark. The transformation $x \mapsto Jf(x)$ can be continuous under any norm. Indeed, we intend to show for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_0 - y\|_X < \delta \Rightarrow \|Jf(x_0) - Jf(y)\| < \varepsilon$. Because $\mathbb{R}^{m \times n}$ is finite-dimensional, any norm (operator norm, supremum norm, ℓ^2 norm) work the same as they are equivalent. Here we use the ℓ^1 norm.

Proof. We know that if the derivative exists we have that

$$f'(x)z = \varphi_V^{-1}\left(Jf(x)\left[z\right]_U\right).$$

Hence we can consider the Fréchet derivative (although we cannot formally write it down...yet):

$$\|\left[f(x_0 + z) - f(x_0)\right] - Jf(x_0)\left[z\right]_U\|_{\ell^1} \stackrel{?}{=} o(\|z\|_X).$$

The ℓ^1 norm can now be represented as

$$\|\cdot\|_{\ell^1} = \sum_{i=1}^m \left| f_i(x_0 + z) - f_i(x_0) - \sum_{j=1}^n \partial_j f_i(x_0) z_j \right|.$$

Define $w_j = \sum_{k=1}^j z_j u_j$, a "partial sum" where $w_n = z$. Thus

$$\|\cdot\|_{\ell^1} = \sum_{i=1}^m \left| \sum_{j=1}^n f_i(x_0 + w_j) - f_i(x_0 + w_{j-1}) - \partial_j f_i(x_0) z_j \right| = \sum_{i=1}^m \left| \sum_{j=1}^n \int_0^{z_j} \frac{\mathrm{d}}{\mathrm{d}s} f_i(x_0 + w_{j-1} + s u_j)\,\mathrm{d}s - \partial_j f_i(x_0) z_j \right|.$$

Here in the second step we applied the fundamental theorem of calculus, and

$$\frac{\mathrm{d}}{\mathrm{d}s} f_i(x_0 + w_{j-1} + s u_j) = \partial_j f_i(x_0 + w_{j-1} + s u_j).$$

Hence by the triangle inequality,

$$\|\cdot\|_{\ell^1} \leqslant \sum_{i=1}^m \sum_{j=1}^n \left| \int_0^{z_j} \partial_j f_i(x_0 + w_{j-1} + s u_j) - \partial_j f_i(x_0)\,\mathrm{d}s \right| \leqslant \varepsilon \|z\|_X,$$

(with some steps omitted) proving the statement as desired. □

Beginning of April 7, 2023

We will continue with the discussion in chain rules. Consider finite-dimensional normed vector spaces X, Y, Z with dimensions n, m, p. Assume further $f : P \to Y$ and $g : Q \to Z$, $P \subset X$ and $Q \subset Y$. Further f differentiable at $x \in P$ and g differentiable at $f(x) \in Q$. We already know from the chain rule that $g \circ f$ is differentiable at x with derivative $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$. Essentially, if X, Y, Z are finite-dimensional vector spaces, then we can replace the derivatives with matrices - the Jacobian matrices are the matrices of transformation of the derivative.

Proposition.

$$J(g \circ f)(x) = Jg(f(x))Jf(x).$$

Remark. Particularly,

$$[J(g \circ f)(x)]_{ij} = \sum_{k=1}^{m} [Jg(f(x))]_{ik}[Jf(x)]_{kj}.$$

This matches the definition of partial derivative in calculus III, where

$$\partial_j h_i(x) = \sum_{k=1}^{m} \partial_k g_i(f(x)) \partial_j f_k(x).$$

We then quickly introduce the gradient vector. Let $f : U \to \mathbb{R}$ be a function, where $U \subset \mathbb{R}^n$ is open. If f is differentiable at $x \in U$, then we can define the gradient vector.

Definition. The **gradient vector** of f at x is evaluated as

$$\nabla f(x) = \begin{bmatrix} \partial_1 f(x) \\ \vdots \\ \partial_n f(x) \end{bmatrix}.$$

Remark. $f'(x)z = z \cdot \nabla f(x)$; and the operator norm $\|f'(x)\|_{\mathbb{R}^n \to \mathbb{R}} = |\nabla f(x)|$.

We have talked about first derivative, so in principle we can discuss more about higher-order derivatives. Indeed, let $f : U \to Y$ where $U \subset X$ is open. For now, assume that f is "nice" enough (in terms of smoothness). From previous lectures we know that $f'(x) \in \mathcal{B}(X, Y)$, so technically taking the derivative again gives

$$f''(x) = (f')'(x) \in \mathcal{B}(X, \mathcal{B}(X, Y)),$$

a linear transformation where the range is the set of bounded transformation $\mathcal{B}(X, Y)$. Specifically,

$$((f''(x))z)x \in Y,$$

where $z, w \in X$. This is because $(f''(x))z$ outputs a linear transformation $\mathcal{B}(X, Y)$.

We can slightly change the notations of the second derivative through a **bilinear transformation**. Specifically, we define $f''(x) = \mathcal{B}^2(X \times X, Y)$.

Definition. Let V, W, Z be real normed vector spaces. A **bilinear transformation** $B : V \times W \to Z$ is a map which is linear in both arguments. Particularly,

$$B(cv_1 + v_2, w) = cB(c_1, w) + B(v_2, w); \quad B(v, cw_1 + w_2) = cB(v, w_1) + B(v, w_2).$$

Remark. Bilinear transformations are *not* linear. Indeed, $B(c(v, w)) = c^2 B(v, w)$.

The bilinear transformation B takes the operator norm

$$\|B\|_{V \times W \to Z} = \sup \frac{\|B(v, w)\|_Z}{\|v\|_V \|w\|_W},$$

where $v \in V \setminus \{0\}$, $w \in W \setminus \{0\}$. B is bounded if $\|B\|$ is finite. Using the same proof, B is continuous if $\|B\|$ is finite. We also define $\mathcal{B}^2(V \times W, Z)$ is the set of all bounded bilinear transformations $B : V \times W \to Z$. Particularly, it is isomorphic to the "ugly definition":

$$\mathcal{B}(V, \mathcal{B}(W, Z)) \cong \mathcal{B}^2(V \times W, Z).$$

Example. Any real inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is a bilinear transformation.

Example. The quadratic form of a matrix $A \in \mathbb{R}^{m \times n}$, defined as $B(x, y) = x \cdot Ay$ is a bilinear transformation $B : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$.

Now back to the double derivative $f'(x_0)(z, w)$, a bilinear transformation from $f : X \to Y$. We have the following identity as a result of the definition:

$$f''(x_0)(z, w) = ((f')'(x_0)z)w.$$

We also have the following claim.

Proposition.

$$f''(x_0)(z, w) = (D_w(D_z f))(x_0).$$

Proof. Consider the difference

$$\frac{D_w f(x_0 + hw) - D_w f(x_0)}{h} - (D_z(f')(x_0))w,$$

where $D_w f(x_0 + hz) = f'(x_0 + hz)w$, and $D_w f(x_0) = f'(x_0)w$. Taking the norms, we have that

$$\|\cdot\|_Y \leqslant \left\| \frac{f'(x_0 + hz) - f'(x_0)}{h} - D_z(f')(x_0) \right\|_{X \to Y} \|w\|_X ,$$

with the operator norm $\|\cdot\|_{X \to Y}$ tending to zero from the derivative of the Fréchet derivative in the direction z. Therefore the above expression $\|\cdot\|_Y$ also goes to zero, proving the claim. $\square$

Beginning of April 10, 2023

Last time we talked about second derivatives of a function, focusing on the second derivative as a linear transformation. Particularly, the following proposition holds:

Proposition.

$$((f')'(x_0)z)w = f''(x_0)(z, w).$$

The second derivative is the space of bounded bilinear transformation, $f''(x_0) \in \mathcal{B}^2(X \times X, Y) \cong \mathcal{B}(X, \mathcal{B}(X, Y))$. This brings the question: is $f''(x_0)$ symmetric? The answer is yes, provided the existence of $f''(x_0)$, and this condition is stronger than the existence of all second partials.

Example. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Here away from the origin,

$$\partial_1 f(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}, \quad \partial_2 f(x,y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2}.$$

Then clearly $\partial_2(\partial_1 f(x,y)) \neq \partial_1(\partial_2 f(x,y))$ at the origin. As f is *not* twice-differentiable, we *cannot* directly switch the order of differentiation!

Restricting the statement to the finite-dimensional case, consider $f : P \to \mathbb{R}$ where P is an open subset of $\mathbb{R}^n$. (We do not lose any information by restricting our co-domain to $\mathbb{R}$ instead of $\mathbb{R}^m$.) Assume $\mathbb{R}^n$ has basis $U = (u_1, \cdots, u_n)$,

$$f''(x_0)(u_j, u_k) = f''(x_0)(u_k, u_j).$$

Particularly,

$$\partial_k \partial_j f(x_0) = \partial_j \partial_k f(x_0).$$

We can make an inductive argument on n-th derivative. Additionally, if $f \in C^k$, then f is k-times differentiable. $f \in C^k$ if and only if mixed k-th partials commute and are continuous.

Definition. The **Hessian matrix** of $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 (with respect to basis U) is defined as

$$Hf(x_0) = \begin{bmatrix} \partial_1\partial_1 & \cdots & \partial_1\partial_n f(x_0) \\ \vdots & \ddots & \vdots \\ \partial_n\partial_1 f(x_0) & \cdots & \partial_n\partial_n f(x_0) \end{bmatrix}.$$

$Hf(x_0)$ is symmetric if $f''(x_0)$ exists and

$$f''(x_0)(z,w) = [z]_U \cdot Hf(x_0)\,[w]_U\,.$$

Remark. Above is an example of a quadratic form of the symmetric matrix $Hf(x_0)$.

We now move forward to Taylor's theorem, introducing the multiindex notation.

Example. Assume $f : \mathbb{R}^3 \to \mathbb{R}$ is six times differentiable at a. We can write

$$\partial_2\partial_1\partial_3\partial_2\partial_2\partial_1 f(a) = \partial_1^2\partial_2^3\partial_3 f(a).$$

We can alternatively write it as $\partial^\alpha f(a)$, where $\alpha = (2,3,1)$.

Definition. A **multiindex** α is an n-tuple of elements that belong to the set $\mathbb{N}_0^n$. It is used to indicate the number of derivatives for each individual component.

Definition. Consider a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n) = \mathbb{N}_0^n$. We will use the following definition/notation:

- We define $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$.
- With $(x_1, \cdots, x_n) \in \mathbb{R}^n$, we define $x^\alpha = (x_1, \cdots, x_n)^{\alpha_1, \cdots, \alpha_n} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
- The absolute value of the multiindex $|\alpha|$ is define as the sum of the indices, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Remark. $|z^\alpha| \leqslant |z|^{|\alpha|}$.

Proposition. (*Counting multiindices*) If $\alpha \in \mathbb{N}_0^n$ is a multiindex of order k, then there are $k!/\alpha!$ distinct tuples of the form $(i_1, \cdots, i_k)$ such that $z_{i_1} \cdots z_{i_k} = z^\alpha$.

Theorem. (*1-d Taylor*) Assume $g \in (C^m[0,1], \mathbb{R})$. Given $x \in (0,1]$, there exists $x^* \in (0, x)$ such that

$$g(x) = \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} x^k + \frac{g^{(m)}(x^*)}{m!} x^m.$$

Proof. Apply the mean value theorem m times. ☐

With the Taylor's theorem in one-dimensional and the multiindex notation, we will now be comfortable in expressing the Taylor's theorem in finite-dimensional spaces. Suppose $f : P \to \mathbb{R}$ and $f \in C^m$, where P is a convex subset of $\mathbb{R}^n$. Suppose given a we wish to approximate $a+z$; we opt to parametrize through $\gamma : [0,1] \to \mathbb{R}^n$ defined by $\gamma(t) = a+tz$. Then we can write

$$f(a+z) = f(\gamma(1)),$$

where $g = f \circ \gamma : [0,1] \to \mathbb{R}$. Then applying Taylor's theorem in one-dimensional case,

$$f(a+z) = g(1) = \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(m)}(t^*)}{m!}.$$

With $g(t) = f(a+tz)$, applying chain rule gives

$$g'(t) = z \cdot \nabla f(a+tz) = \sum_{i_1=1}^{n} z_{i_1} \partial_{i_1} f(\gamma(t)); \quad g^{(k)}(t) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} z_{i_1} \cdots z_{i_k} \partial_{i_1} \cdots \partial_{i_k} f(\gamma(t)).$$

We can then apply the multiindex to return a more beautiful expression. A bit to cover in the next lecture.

Beginning of April 12, 2023

Last time we were talking about Taylor's theorem in multiple dimensions. Specifically, $f : U \to \mathbb{R}$ where U is an open convex subset in $\mathbb{R}^n$. With $f \in C^m(U; \mathbb{R})$, we have the composite function for estimating a point $a+z$ based on a point a:

$$g = f \circ \gamma : [0,1] \to \mathbb{R}, \quad \gamma(t) = a + tz.$$

Here the γ walks along the segment from a to $a+z$. With

$$g'(t) = z \cdot \nabla f(\gamma(t)) = \sum_{i_i=1}^{n} z_{i_1} \partial_{i_1} f(\gamma(t)), \quad g^{(k)}(t) = \sum_{i_i=1}^{n} \cdots \sum_{i_k=1}^{n} z_{i_1} z_{i_k} \partial_{i_1} \cdots \partial_{i_k} f(\gamma(t)),$$

the single-variable Taylor theorem concludes that

$$f(a+z) = g(1) = \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(m)}(t^*)}{m!}$$

for some $t^* \in (0,1)$.

Remark. The notation for $g^{(k)}(t)$ is obtained from reproducing the chain rule k times. Using multiindex notation, we use the counting proposition to count for repetitive factors. Summing over the set where $|\alpha| = k$,

$$g^{(k)}(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha \partial^\alpha f(\gamma(t)).$$

Therefore

$$f(a+z) = \sum_{k=0}^{m-1} \sum_{|\alpha|=k} \frac{z^\alpha \partial^\alpha f(a)}{\alpha!} + \sum_{|\alpha|=m} \frac{z^\alpha \partial^\alpha f(z^*)}{\alpha!}.$$

Remark. The last term is $o(|z|^m)$.

Then we move on to implicit function theorem (in a more simple way first). Recall the vertical line test - consider an example $y = \sqrt{1-x^2}$, or the upper unit circle. At the point $x = 1$, we cannot extend the unit circle to the negative y-axis unless we write some $x = \sqrt{1-y^2}$; but we cannot extend the unit circle to the negative x-axis this way. Closely speaking, the unit circle is the solution set of

$$F(x,y) = x^2 + y^2$$

with the graph of the circle the level set $F^{-1}(1)$. If we restrict the domain to $y > 0$, we can write the solution set as $y = \sqrt{1-x^2}$; and we are interested in the property that goes further than the vertical line test. More specifically, we want to test the solution $y = g(x)$ only based on the function $F(x,y) = x^2 + y^2$.

Theorem. (*Implicit function theorem in the plane*) Let U be an open subset of $\mathbb{R}^2$. Assume $F \in C^r(U;\mathbb{R})$ for some $r \in \mathbb{N}$. Let (x_0, y_0) be a point in U such that $\partial_2 F(x_0,y_0) \neq 0$. Denote $z_0 = F(x_0,y_0)$. Then there exists a unique function $g : (x_0 - \delta, x_0 + \delta) \to (y_0 - \varepsilon, y_0 + \varepsilon)$ such that

$$\text{Gragh } g = F^{-1}(z_0) \cap ((x_0 - \delta, x_0 + \delta) \times (y_0 - \varepsilon, y_0 + \varepsilon)), \quad g \in C^r((x_0 - \delta, x_0 + \delta)).$$

Here the limitation of the co-domain determines the uniqueness of the function. Additionally note that the implicit function theorem only provides a (perhaps small) neighborhood of x_0; if we can apply the implicit function theorem on all of x_0 in the domain, we can get a global representation. Sometimes it would be more convenient to calculate for the set $(x_0, y_0) : \partial_2 f(x_0, y_0) \neq 0$.

Example. Consider the example

$$G(x,y) = (x+y)\cos xy.$$

Additionally,

$$(0,1) \in G^{-1}(1) = \{(x,y) : G(x,y) = 1\}.$$

Can we write $G^{-1}(1)$ as $\{(x,y) : y = g(x)\}$ in some neighborhood of $(0,1)$?

Sol. Compute

$$\partial_2 G(x,y) = \cos xy - x(x+y)\sin xy \Rightarrow \partial_2 G(0,1) = 1 \neq 0,$$

hence the tl;dr answer is *yes*.

Beginning of April 14, 2023

Today we will first talk a bit about contraction mapping principle (or the Banach fixed point theorem), then we will continue on implicit function theorem.

Definition. $x \in X \cap Y$ is a **fixed point** of $f : X \to Y$ if $f(x) = x$.

Definition. Let (X, d) be a metric space. $\varphi : X \to X$ is a **contraction** on X if there exists $c \in [0,1)$ such that

$$d(\varphi(x), \varphi(y)) \leqslant cd(x, y)$$

for every $x, y \in X$. That is, the distance between the two points *shrink*; the function decreases distances.

Remark. Contractions are uniformly continuous.

Theorem. (*Banach fixed point theorem*) Let (X, d) be a complete metric space. Let $\varphi : X \to X$ be a contraction. Then φ has exactly one fixed point.

Proof. (Uniqueness) assume $\varphi(x) = x$ and $\varphi(y) = y$. Then

$$d(x, y) = d(\varphi(x), \varphi(y)) \leqslant cd(x, y) \Rightarrow (1 - c)d(x, y) \leqslant 0.$$

As $c < 1$ and $d(x, y)$ is nonnegative, the only possibility is that $d(x, y) = 0$.
(Existence) choose $x_0 \in X$, define $x_n = \varphi^n(x_0)$. It suffices to show that $(x_n)_n$ is Cauchy. We then have

$$d(x_m, x_n) \leqslant \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leqslant \sum_{k=n}^{m-1} c^k d(x_1, x_0) = c^n d(x_1, x_0) \sum_{j=0}^{m-n-1} c^j \overset{n\to\infty}{\to} 0.$$

This proves the claim. □

We then discuss the more general form of the implicit function theorem. (Note: this is *not* the most implicit function theorem. The more general form takes the form of a Banach space, but we will stick with Euclidean spaces for now.)

We will use the following notation for the implicit function theorem. Specifically, we are interested in a mapping from $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, specifically $F : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n \times \mathbb{R}^m$. The elements of U takes form of $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, where $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_m)$. The Jacobian matrix takes

$$JF(\mathbf{x_0}, \mathbf{y_0}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{bmatrix} = \begin{bmatrix} J_1F(\mathbf{x_0}, \mathbf{y_0}) & J_2F(\mathbf{x_0}, \mathbf{y_0}) \end{bmatrix}.$$

Theorem. (*Implicit function theorem*) Let U be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ containing $(\mathbf{x_0}, \mathbf{y_0})$. Assume $F \in C^k(U, \mathbb{R}^m)$ for some $k \in \mathbb{N}$. Write $\mathbf{z_0} = F(\mathbf{x_0}, \mathbf{y_0})$. Assume $J_2F(\mathbf{x_0}, \mathbf{y_0})$ is invertible; then there exists a neighborhood V of $\mathbf{x_0}$ and W of $\mathbf{y_0}$, with $V \times W \subset U$, and a unique function $g : V \to W$ such that $F(\mathbf{x}, g(\mathbf{x})) = \mathbf{z_0}$ for every $x \in V$ ($v = g(x)$ is locally the level set of $F^{-1}(\mathbf{z_0})$), and $g \in C^k(V; W)$.

Beginning of March 17, 2023

Today we will prove the general statement of the implicit function theorem.

Theorem. (*Implicit function theorem*) Let U be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ containing $(\mathbf{x_0}, \mathbf{y_0})$. Assume $F \in C^k(U, \mathbb{R}^m)$ for some $k \in \mathbb{N}$. Write $\mathbf{z_0} = F(\mathbf{x_0}, \mathbf{y_0})$. Assume $K_2F(\mathbf{x_0}, \mathbf{y_0})$ is invertible; then there exists a neighborhood V of $\mathbf{x_0}$ and W of $\mathbf{y_0}$, with $V \times W \subset U$, and a unique function $g : V \to W$, such that

- $F(\mathbf{x}, g(\mathbf{x})) = \mathbf{z_0}$ for every $\mathbf{x} \in V$, and $\operatorname{graph} g = F^{-1}(\mathbf{z_0}) \cap (V \times W)$,
- $Jg(x) = J_2F(x, g(x))^{-1} J_1F(x, g(x))$ for every $x \in V$,
- $g \in C^k(V; W)$.

Remark. TLDR: on the set S where $F = \mathbf{z_0}$, we can solve for $\mathbf{y}$ as a function of $\mathbf{x}$ near $(\mathbf{x_0}, \mathbf{y_0}) \in S$, provided that $J_2F(\mathbf{x_0}, \mathbf{y_0})$ is invertible.

Proof. Without loss of generality, let $(\mathbf{x_0}, \mathbf{y_0}) = (\mathbf{0}, \mathbf{0}) \in \mathbb{R}^n \times \mathbb{R}^m$; $\mathbf{z_0} = \mathbf{0} \in \mathbb{R}^m$. We want to find $g(\mathbf{x})$, defined near $\mathbf{0} \in \mathbb{R}^n$, such that $F(\mathbf{x}, g(\mathbf{x})) = \mathbf{0}$ on $V = \operatorname{dom} g$. First, for $\mathbf{x}, \mathbf{y}$ in the neighborhood of interest, by the derivative we have that

$$F(\mathbf{x}, \mathbf{y}) = F(\mathbf{0}, \mathbf{0}) + F'(\mathbf{0}, \mathbf{0})(\mathbf{x}, \mathbf{y}) + R(\mathbf{x}, \mathbf{y}) = 0 + J_1F(\mathbf{0}, \mathbf{0})\mathbf{x} + J_2F(\mathbf{0}, \mathbf{0})\mathbf{y} + R(\mathbf{x}, \mathbf{y}).$$

Especially for the second step we have that

$$JF(\mathbf{0}, \mathbf{0})(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} \underbrace{J_1F(\mathbf{0}, \mathbf{0})}_{\in \mathbb{R}^{m \times n}} & \underbrace{J_2F(\mathbf{0}, \mathbf{0})}_{\in \mathbb{R}^{m \times m}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} J_1F(\mathbf{0}, \mathbf{0}) & J_2F(\mathbf{0}, \mathbf{0}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}.$$

Additionally, the remainder term satisfies $R(\mathbf{x}, \mathbf{y}) = o(|(\mathbf{x}, \mathbf{y})|)$ as $(\mathbf{x}, \mathbf{y}) \to (\mathbf{0}, \mathbf{0})$.
Solving for the equation, we have that

$$F(\mathbf{x}, \mathbf{y}) = \mathbf{0} \Rightarrow \mathbf{y} = -J_2F(\mathbf{0}, \mathbf{0})^{-1} \left[J_1F(\mathbf{0}, \mathbf{0})x + R(\mathbf{x}, \mathbf{y}) \right],$$

and note that $J_2F(\mathbf{0}, \mathbf{0})$ is invertible by assumption. The above equation *almost* solves for y as a function of x, but it's not quite there. However, we can define

$$K_{\mathbf{x}}(\mathbf{y}) = -J_2F(\mathbf{0}, \mathbf{0})^{-1} \left[J_1F(\mathbf{0}, \mathbf{0})x + R(\mathbf{x}, \mathbf{y}) \right].$$

Given $\mathbf{x}$ close to $\mathbf{0} \in \mathbb{R}^n$, we look for $\mathbf{y}$ such that $K_{\mathbf{x}}(\mathbf{y}) = \mathbf{y}$. The idea is to find $r > 0$ and $\tau > 0$ such that $K_{\mathbf{x}}$ is a contraction on $\overline{W} = \overline{B(\mathbf{0}, r)}$ whenever $\mathbf{x} \in B(\mathbf{0}, \tau) = V$. Then we can define

$$g(\mathbf{x}) = [\text{fixed point of } K_{\mathbf{x}}].$$

Then *when* is $K_{\mathbf{x}} = -J_2F(\mathbf{0},\mathbf{0})^{-1}[J_1F(\mathbf{0},\mathbf{0})\mathbf{x} + R(\mathbf{x}, \cdot)]$ a contraction? We want to show that $K_{\mathbf{x}}$ does decrease norm in an uniform way; then we need to show that $K_{\mathbf{x}}$ does map $\overline{W} \to \overline{W}$.
Regarding the norm-decreasing property,

$$K_{\mathbf{x}}(\mathbf{y_1}) - K_{\mathbf{x}}(\mathbf{y_2}) = -J_2F(\mathbf{0},\mathbf{0})^{-1}[J_1F(\mathbf{0},\mathbf{0})x + R(\mathbf{x}, \mathbf{y_1})] + J_2F(\mathbf{0},\mathbf{0})^{-1}[J_1F(\mathbf{0},\mathbf{0})\mathbf{x} + R(\mathbf{x}, \mathbf{y_2})].$$

As they share a common term that cancel out, we can simplify the expression as

$$K_{\mathbf{x}}(\mathbf{y_1}) - K_{\mathbf{x}}(\mathbf{y_2}) = J_2F(\mathbf{0},\mathbf{0})^{-1}[R(\mathbf{x},\mathbf{y_2}) - R(\mathbf{x},\mathbf{y_1})] = J_2F(\mathbf{0},\mathbf{0})^{-1}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}R(\mathbf{x}, (1-t)\mathbf{y_1} + \mathbf{y_2})(y_2 - y_1)\,\mathrm{d}t.$$

On the other hand, using the chain rule, we obtain

$$K_{\mathbf{x}}(\mathbf{y_1}) - K_{\mathbf{x}}(\mathbf{y_2}) = J_2F(\mathbf{0},\mathbf{0})^{-1}\int_0^1 J_2F(\mathbf{x}, (1-t)\mathbf{y_1} + t\mathbf{y_2})(y_2 - y_1)\,\mathrm{d}t.$$

This implies

$$\|K_{\mathbf{x}}(\mathbf{y_1}) - K_{\mathbf{x}}(\mathbf{y_2})\| \leqslant \left\|J_2F(\mathbf{0},\mathbf{0})^{-1}\right\| \int_0^1 \|J_2R(\mathbf{x}, (1-t)\mathbf{y_1} + t\mathbf{y_2})\| \, |\mathbf{y_2} - \mathbf{y_1}|\,\mathrm{d}t$$

Then we can choose $r > 0$ such that $\max\{|\mathbf{x}|, |\mathbf{y}|\} < r$ implies

$$\left\|J_2F(\mathbf{0},\mathbf{0}^{-1}\right\| \cdot \|J_2R(\mathbf{x},\mathbf{y})\| \leqslant \frac{1}{2} \Rightarrow |K_{\mathbf{x}}(\mathbf{y_1}) - K_{\mathbf{x}}(\mathbf{y_2})| \leqslant \frac{1}{2}|\mathbf{y_1} - \mathbf{y_2}|.$$

Then we still need to prove that the image is the subset of the space of domain. Particularly, we can make sure that $|K_{\mathbf{x}}(0)| < r/2$, which implies

$$|K_{\mathbf{x}}(\mathbf{y})| \leqslant |K_{\mathbf{x}}(\mathbf{y}) - K_{\mathbf{x}}(\mathbf{0})| + |K_{\mathbf{x}}(\mathbf{0})| < \frac{1}{2}|\mathbf{y} - \mathbf{0}| + \frac{r}{2} \leqslant r.$$

Choosing $\tau \in (0, r)$ such that $|\mathbf{x}| < \tau \Rightarrow |K_{\mathbf{x}}(0)| < r/2$ suffices. Therefore $K_{\mathbf{x}} : \overline{B(0,r)} \to \overline{B(0,r)}$ is a contraction whenever $|\mathbf{x}| < \tau$. Then by the contraction mapping principle, we have a unique fixed point.
The next step is to show that g is differentiable at $\mathbf{0} \in \mathbb{R}^n$ with $Jg(\mathbf{0}) = -J_2F(\mathbf{0},\mathbf{0})^{-1}J_1F(\mathbf{0},\mathbf{0})$.

$$g(\mathbf{x}) = K_{\mathbf{x}}(g(\mathbf{x})) = -J_2F(\mathbf{0},\mathbf{0})^{-1}[J_1F(\mathbf{0},\mathbf{0})\mathbf{x} + R(\mathbf{x}, g(\mathbf{x})].$$

A manipulation of terms give

$$\frac{g(\mathbf{x}) - g(\mathbf{0}) - (-J_2(\mathbf{0},\mathbf{0})^{-1}J_1F(\mathbf{0},\mathbf{0})\mathbf{x})}{\|\mathbf{x}\|} = J_2F(\mathbf{0},\mathbf{0})^{-1}\frac{R(\mathbf{x}, g(\mathbf{x}))}{\|\mathbf{x}\|}.$$

It suffices to show that $R(\mathbf{x}, g(\mathbf{x}))/\|\mathbf{x}\| \to 0$ as $\mathbf{x} \to \mathbf{0}$. But also note that

$$\frac{R(\mathbf{x}, g(\mathbf{x}))}{\|x\|} = \frac{R(\mathbf{x}, g(\mathbf{x}))}{|(\mathbf{x}, g(\mathbf{x}))|} \cdot \frac{|(\mathbf{x}, g(\mathbf{x}))|}{\|\mathbf{x}\|}.$$

If we can show that g is *Lipschitz*, then the first component goes to zero as $\mathbf{x} \to 0$, as $\mathbf{x} \to 0$ implies $(\mathbf{x}, g(\mathbf{x})) \to (\mathbf{0},\mathbf{0})$, whereas the second term remains bounded. Then it suffices to show that $|g(\mathbf{x})| \leqslant L\|\mathbf{x}\|$ for some $L \geqslant 0$

around zero. We thus have

$$
\begin{aligned}
|g(\mathbf{x})| &= |K_{\mathbf{x}}(g(\mathbf{x}))| \\
&= |K_{\mathbf{x}}(g(\mathbf{x})) - K_{\mathbf{x}}(\mathbf{0})| + |K_{\mathbf{x}}(\mathbf{0})| \\
&\leqslant \frac{1}{2}\,|g(\mathbf{x}) - \mathbf{0}| + \left|J_2F(\mathbf{0},\mathbf{0})^{-1}(J_1F(\mathbf{0},\mathbf{0})\mathbf{x} + R(\mathbf{x},\mathbf{0})\right| \\
&\leqslant \frac{1}{2}\,|g(\mathbf{x})| + \left[\left\|J_2F(\mathbf{0},\mathbf{0})^{-1}\right\|\,\|J_1F(\mathbf{0},\mathbf{0})\| + \frac{\|R(\mathbf{x},\mathbf{0})\|}{\|\mathbf{x}\|}\right]\|\mathbf{x}\|\,,
\end{aligned}
$$

whereas the bracketed term is bounded above by $L/2$. This proves the remainder term is $o(\|\mathbf{x}\|)$, hence the Jacobian is then

$$J = -J_2(\mathbf{0},\mathbf{0})^{-1}J_1F(\mathbf{0},\mathbf{0})\mathbf{x}.$$

The last step is to obtain the formula to obtain the formula for the Jacobian evaluated at not just zero but some point near zero. To do so, J_2F is continuous at $(\mathbf{0},\mathbf{0}) \in \mathbb{R}^n \times \mathbb{R}^m$ and g is continuous at $\mathbf{0} \in \mathbb{R}^n$. Additionally, $J_2F(\mathbf{0},\mathbf{0})$ is invertible, and $GL(m)$, the set of invertible matrices, is an open subset of $\mathbb{R}^{m\times m}$. Therefore $J_2F(\mathbf{x},g(\mathbf{x}))$ is invertible for small enough $\mathbf{x}$. Running similar argument as step 2 gives the expression

$$Jg(\mathbf{x}) = J_2F(\mathbf{x},g(\mathbf{x}))^{-1}J_1F(\mathbf{x},g(\mathbf{x})),$$

and that g is C^1; by induction we can extend it to C^k. □

Beginning of April 19, 2023

Today we will talk about the inverse function theorem and introduce the differential forms.

Theorem. (*Inverse function theorem*) Assume $\mathbf{x_0} \in U$, and U is an open subset in $\mathbb{R}^m$. Additionally, f maps U to $\mathbb{R}^m$, and is C^k for some $k \in \mathbb{N}$, $\mathbf{y_0} = f(\mathbf{x_0})$, and $Jf(\mathbf{x_0})$ is invertible. Then there exists neighborhoods V including $\mathbf{x_0}$ and W including $\mathbf{y_0}$, and a unique function $g : W \to V$ such that

$$g = (f\,|_{V\to W})^{-1}\,,$$

and g is a C^k mapping from W to V. Particularly, $g(f(x)) = x$ for $x \in V$ and $f(g(y)) = y$ for $y \in W$.

Proof. We hope to look at the graph of f, defined as

$$\text{graph }(f) = \{(\mathbf{x},\mathbf{y}) \in U \times \mathbb{R}^m : f(\mathbf{x}) - \mathbf{y} = \mathbf{0}\} =: F(\mathbf{x},\mathbf{y}) \in \mathbb{R}^m.$$

Defining $F : U \times \mathbb{R}^m \to \mathbb{R}^m$, the graph of f is the level set of F at zero, or

$$\text{graph }(f) = F^{-1}(\mathbf{0}).$$

We then use the implicit function theorem on F. Particularly, we have $\mathbf{y} = f(\mathbf{x})$ on $F^{-1}(\mathbf{0})$; and we want to invert this to obtain $\mathbf{x} = f^{-1}(\mathbf{y})$ near $(\mathbf{x_0},\mathbf{y_0})$.

Looking at $J_1F(\mathbf{x_0},\mathbf{y_0}) = Jf(\mathbf{x_0})$ as the Jacobian with respect to that $\mathbf{x}$ variables, the Jacobian is invertible by assumption, therefore there exists neighborhoods V including $\mathbf{x_0}$ and W including $\mathbf{y_0}$ and a C^k function $g : W \to V$ such that $F(g(\mathbf{y},\mathbf{y}) = 0$ for every $\mathbf{y} \in W$. This shows that g is a right inverse for f, in that

$$\mathbf{0} = F(g(\mathbf{y}),\mathbf{y}) = f(g(\mathbf{y})) - \mathbf{y}$$

for all $\mathbf{y} \in W$. This shows that g is a right inverse for f on the restricted domain. It remains to show that g is a left inverse for f. We apply the implicit function theorem again. Now consider $G(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}) - \mathbf{x}$ on $G^{-1}(\mathbf{0})$. We note that $G(\mathbf{x_0}, \mathbf{y_0}) = G(g(\mathbf{y_0}, \mathbf{y_0}) = \mathbf{0}$, so $(\mathbf{x_0}, \mathbf{y_0}) \in G^{-1}(\mathbf{0})$; additionally $J_2 G(\mathbf{x_0}, \mathbf{y_0}) = Jg(\mathbf{y_0}) = Jf(\mathbf{x_0})^{-1}$, as $f \circ g = \mathrm{id}_W \Rightarrow Jf(g(y_0))Jg(y_0) = I_{mxm}$. Therefore there exists neighborhoods $\tilde{V}$ including $\mathbf{x_0}$ and $\tilde{W}$ including $\mathbf{y_0}$ and $\tilde{f} \in C^k(\tilde{V}; \tilde{W})$ such that for every $\mathbf{x} \in \tilde{V}$ we have $\mathbf{0} = G(\mathbf{x}, \tilde{f}(x)) = g(\tilde{f}(\mathbf{x})) - \mathbf{x} \Rightarrow g \circ \tilde{f} = \mathrm{id}_{\tilde{V}}$. Furthermore, as the two inverses must align, the proof is complete. ☐

Remark. There is no way to turn this "local" inverse function theorem into a global one.

Example. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}.$$

We have that

$$Jf(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix},$$

which is invertible for all $(x, y) \in \mathbb{R}^2$; but $f(x, y + 2\pi) f(x, y)$ for every $(x, y) \in \mathbb{R}^2$, so there does not exist a global inverse.

And that would be the end of this set of lecture notes. In fact, Professor Leslie did *not* really give an in-depth intro on the topic of differential forms, although it was arguably one of the most important sections. (He did talk about it, but it wasn't tested anyways so I didn't really bother to take notes.) Nevertheless, his "lecture notes" follows directly from the differential forms section of Pugh's *Real Mathematical Analysis* book, so an interested reader can instead redirect to Pugh.