Homework 2

2.1 - 11, 12

For Problems 10–12, determine tr(A) for the given matrix.

10. $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

11. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & -2 \\ 7 & 5 & -3 \end{bmatrix}$.

12. $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 0 & 1 & -5 \end{bmatrix}$.

11. $\mathrm{Tr}(A) = 1+2-3 = 0$

12. $\mathrm{Tr}(A) = 2+2-5 = -1$

2.2 - 7, 8, 11a, 13, 39, 40

For Problems 6–9, determine $A\mathbf{c}$ by computing an appropriate linear combination of the column vectors of A.

6. $A = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

7. $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 5 \\ 7 & -6 & 3 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

8. $A = \begin{bmatrix} -1 & 2 \\ 4 & 7 \\ 5 & -4 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

9. $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}, \mathbf{c} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$.

7. $Ac = \begin{bmatrix} [3\ {-1}\ 4]\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \\ [2\ 1\ 5]\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \\ [7\ {-6}\ 3]\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -13 \\ -13 \\ -16 \end{bmatrix}$

8. $Ac = \begin{bmatrix} [-1\ 2]\begin{bmatrix} 5 \\ -1 \end{bmatrix} \\ [4\ 7]\begin{bmatrix} 5 \\ -1 \end{bmatrix} \\ [5\ {-4}]\begin{bmatrix} 5 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \\ 29 \end{bmatrix}$

11. Find A^2, A^3, and A^4 if

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix}$.

11(a). $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$= \left[\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right]$

$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$

$A^3 = AAA = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$= \left[\begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right]$

$= \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}$

$A^4 = AAAA = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$= \left[\begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right]$

$= \begin{bmatrix} -31 & -24 \\ 48 & 17 \end{bmatrix}$

13. If $A = \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix}$, calculate A^2 and verify that A satisfies $A^2 + 4A + 18I_2 = 0_2$.

$A^2 = \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix} = \left[\begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} -5 \\ -6 \end{bmatrix}\right]$

$= \begin{bmatrix} -26 & 20 \\ -24 & 6 \end{bmatrix}$

$\begin{bmatrix} -26 & 20 \\ -24 & 6 \end{bmatrix} + 4\begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$

39. If A and B are $n \times n$ symmetric matrices such that $AB = BA$, then AB is symmetric.

40. Use the properties of the transpose to prove that

(a) AA^T is a symmetric matrix.

(b) $(ABC)^T = C^T B^T A^T$.

39. $A = A^T$, $B = B^T$.

$(AB)^T = B^T A^T = BA = AB$

$(AB)^T = AB$, so AB is symmetric.

40. $(AA^T)^T = (A^T)^T A^T = AA^T$

$(AA^T)^T = AA^T$, so AA^T is symmetric

$(ABC)^T = C^T (AB)^T = C^T B^T A^T$.

2.4 - 11, 13, 21

For Problems 9–18, use elementary row operations to reduce the given matrix to row-echelon form, and hence determine the rank of each matrix.

9. $\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$.

10. $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$.

11. $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix}$.

13. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

11. $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 3 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

1 $A_{12}(-1)$ 2 $A_{13}(-3)$ 3 $A_{23}(4)$ Rank 2

13. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 1 & -2 \\ 2 & -1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 2 & -1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

$\overset{4}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & -6 & 11 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 23 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ Rank 3

1 P_{12} 2 $A_{21}(-1)$ 3 $A_{32}(-1)$ 4 $A_{13}(-2)$ 5 $A_{23}(6)$ 6 $M_3(\frac{1}{23})$

For Problems 19–26, reduce the given matrix to reduced row-echelon form and hence determine the rank of each matrix.

19. $\begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$.

20. $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$.

21. $\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

22. $\begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \\ 6 & -6 & 12 \end{bmatrix}$.

$\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 11 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 11 \\ 0 & -1 & -3 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 3 \\ 0 & -2 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ Rank = 3

2.5 - 1, 4, 7, 24, 34, 38

For Problems 1–12, use Gaussian elimination to determine the solution set to the given system.

1. $x_1 - 5x_2 = 3,$
$3x_1 - 9x_2 = 15.$

2. $4x_1 - x_2 = 8,$
$2x_1 + x_2 = 1.$

3. $7x_1 - 3x_2 = 5,$
$14x_1 - 6x_2 = 10.$

4. $x_1 + 2x_2 + x_3 = 1,$
$3x_1 + 5x_2 + x_3 = 3,$
$2x_1 + 6x_2 + 7x_3 = 1.$

5. $3x_1 - x_2 = 1,$
$2x_1 + x_2 + 5x_3 = 4,$
$7x_1 - 5x_2 - 8x_3 = -3.$

6. $3x_1 + 5x_2 - x_3 = 14,$
$x_1 + 2x_2 + x_3 = 3,$
$2x_1 + 5x_2 + 6x_3 = 2.$

7. $6x_1 - 3x_2 + 3x_3 = 12,$
$2x_1 - x_2 + x_3 = 4,$
$-4x_1 + 2x_2 - 2x_3 = -8.$

8. $2x_1 - x_2 + 3x_3 = 14,$
$3x_1 + x_2 - 2x_3 = -1,$
$7x_1 + 2x_2 - 3x_3 = 3,$
$5x_1 - x_2 - 2x_3 = 5.$

9. $2x_1 - x_2 - 4x_3 = 5,$
$3x_1 + 2x_2 - 5x_3 = 8,$
$5x_1 + 6x_2 - 6x_3 = 20,$
$x_1 + x_2 - 3x_3 = -3.$

10. $x_1 + 2x_2 - x_3 + x_4 = 1,$
$2x_1 + 4x_2 - 2x_3 + 2x_4 = 2,$
$5x_1 + 10x_2 - 5x_3 + 5x_4 = 5.$

1. $\left[\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & -9 & 15 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 1 & -3 & 5 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & 1 \end{array}\right]$

$x_2 = 1$, $x_1 = 3 + 5 = 8$

4. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & -1 & -6 & 2 \\ 2 & 6 & 7 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & -7 & 1 \\ 2 & 6 & 7 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{7}{3} & -\frac{1}{3} \\ 0 & 2 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{7}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{7}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 \end{array}\right]$

$x_3 = -1$, $x_2 = 2$, $x_1 = -2$

7. $\left[\begin{array}{ccc|c} 6 & -3 & 3 & 12 \\ 2 & -1 & 1 & 4 \\ -4 & 2 & -2 & -8 \end{array}\right] \sim \begin{bmatrix} 2 & -1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The solution has infinte number of solutions

2.6 - 2, 5, 11, 19, 21, 25, 27, 29

24. Determine all values of the constant k for which the following system has **(a)** no solution, **(b)** an infinite number of solutions, and **(c)** a unique solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3, \\ 2x_1 + 5x_2 + x_3 &= 7, \\ x_1 + x_2 - k^2 x_3 &= -k. \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^2 & k \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & 1-k^2 & -3-k \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4-k^2 & -2-k \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ 1 & 1 & -k^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & -1 \\ 1 & 1 & -k^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & k^2-1 \\ 1 & 1 & -k^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & k^2-1 \\ 0 & -2 & -k^2-2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & k^2-1 \\ 0 & 0 & k^2-4 \end{bmatrix}$$

No solution: $k = 2$

Unique Solution: $k \neq \pm 2$

Infinitely many solutions: $k = 2$

34. **(a)** An $n \times n$ system of linear equations whose matrix of coefficients is a lower triangular matrix is called a **lower triangular system**. Assuming that $a_{ii} \neq 0$ for each i, devise a method for solving such a system that is analogous to the back substitution method.

(b) Use your method from (a) to solve

$$\begin{aligned} x_1 &= 2, \\ 2x_1 - 3x_2 &= 1, \\ 3x_1 + x_2 - x_3 &= 8. \end{aligned}$$

(a) Solve for x_1 from 1^{st} row,

then put x_1 into 2^{nd} row to find x_2,

so on.

(b) $x_1 = 2$

$x_2 = \frac{4-1}{3} = 1$

$x_3 = \frac{7-8}{-1} = 1.$

$x_1, x_2, x_3 = (2, 1, 1).$

For Problems 36–46, determine the solution set to the given system.

36. $\begin{aligned} 3x_1 + 2x_2 - x_3 &= 0, \\ 2x_1 + x_2 + x_3 &= 0, \\ 5x_1 - 4x_2 + x_3 &= 0. \end{aligned}$

37. $\begin{aligned} 2x_1 + x_2 - x_3 &= 0, \\ 3x_1 - x_2 + 2x_3 &= 0, \\ x_1 - x_2 - x_3 &= 0, \\ 5x_1 + 2x_2 - 2x_3 &= 0. \end{aligned}$

38. $\begin{aligned} 2x_1 - x_2 - x_3 &= 0, \\ 5x_1 - x_2 + 2x_3 &= 0, \\ x_1 + x_2 + 4x_3 &= 0. \end{aligned}$

39. $\begin{aligned} (1+2i)\,x_1 + (1-i)x_2 + x_3 &= 0, \\ ix_1 + (1+i)x_2 - ix_3 &= 0, \\ 2ix_1 + x_2 + (1+3i)x_3 &= 0. \end{aligned}$

40. $\begin{aligned} 3x_1 + x_2 + x_3 &= 0, \\ 6x_1 - x_2 + 2x_3 &= 0, \\ 12x_1 + 6x_2 + 4x_3 &= 0. \end{aligned}$

38. $\begin{bmatrix} 2 & -1 & -1 \\ 5 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$r^\# < n$, so there are infinitely many solutions

$x_1, x_2, x_3 = \{(-t, -3t, t)\}$

For Problems 1–4, verify by direct multiplication that the given matrices are inverses of one another.

1. $A = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}, A^{-1} = \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix}.$

2. $A = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}.$

3. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$ provided $ad - bc \neq 0$.

4. $A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}, A^{-1} = \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix}.$

2. $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

For Problems 5–18, determine A^{-1}, if possible, using the Gauss-Jordan method. If A^{-1} exists, check your answer by verifying that $AA^{-1} = I_n$.

5. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$

6. $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}.$

7. $A = \begin{bmatrix} 1 & -i \\ -1+i & 2 \end{bmatrix}.$

8. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

5. $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$

$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 11 \\ 4 & -3 & 10 \end{bmatrix}.$

10. $A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}.$

11. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$

11. $\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank = 2, inverse doesn't exist.

19. Let $A = \begin{bmatrix} -1 & -2 & 3 \\ -1 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix}$. Find the third column vector of A^{-1} without determining the other columns of the inverse matrix.

$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3^{rd} column vector: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$-x - 2y + 3z = 0$

$-x + y + z = 0$

$-x - 2y - z = 1$

$\Rightarrow \left[\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -2 & -1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & -4 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 1 & -1/4 \end{bmatrix}$

$x = -\frac{5}{12}$ $y = -\frac{1}{6}$ $z = -\frac{1}{4}$

For Problems 21–26, use A^{-1} to find the solution to the given system.

21. $\begin{aligned} 6x_1 + 20x_2 &= -8, \\ 2x_1 + 7x_2 &= 2. \end{aligned}$

22. $\begin{aligned} x_1 + 3x_2 &= 1, \\ 2x_1 + 5x_2 &= 3. \end{aligned}$

23. $\begin{aligned} x_1 + x_2 - 2x_3 &= -2, \\ x_2 + x_3 &= 3, \\ 2x_1 + 4x_2 - 3x_3 &= 1. \end{aligned}$

24. $\begin{aligned} x_1 - 2ix_2 &= 2, \\ (2-i)x_1 + 4ix_2 &= -i. \end{aligned}$

25. $\begin{aligned} 3x_1 + 4x_2 + 5x_3 &= 1, \\ 2x_1 + 10x_2 + x_3 &= 1, \\ 4x_1 + x_2 + 8x_3 &= 1. \end{aligned}$

26. $\begin{aligned} x_1 + x_2 + 2x_3 &= 12, \\ x_1 + 2x_2 - x_3 &= 24, \\ 2x_1 - x_2 + x_3 &= -36. \end{aligned}$

21. $Ax = \begin{bmatrix} -8 \\ 2 \end{bmatrix} \to x = A^{-1}\begin{bmatrix} -8 \\ 2 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 6 & 20 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & \frac{10}{3} & \frac{1}{6} & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{10}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 7/2 & -10 \\ 0 & 1 & -1 & 3 \end{array}\right]$

$x = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -48 \\ 14 \end{bmatrix}$

$x_1 = -48, x_2 = 14.$

25. $x = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\left[\begin{array}{ccc|ccc} 3 & 4 & 5 & 1 & 0 & 0 \\ 2 & 10 & 1 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{5}{3} & \frac{1}{3} & 0 & 0 \\ 2 & 10 & 1 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 4/3 & 5/3 & 1/3 & 0 & 0 \\ 0 & 1 & -7/22 & -1/11 & 3/22 & 0 \\ 0 & -13/3 & 4/3 & -4/3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4/3 & 5/3 & 1/3 & 0 & 0 \\ 0 & 1 & -7/22 & -1/11 & 3/22 & 0 \\ 0 & -13/3 & 4/3 & -4/3 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 23/11 & 5/11 & -2/11 & 0 \\ 0 & 1 & -7/22 & -1/11 & 3/22 & 0 \\ 0 & 0 & -1/22 & -19/11 & 13/22 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 23/11 & 5/11 & -2/11 & 0 \\ 0 & 1 & -7/22 & -1/11 & 3/22 & 0 \\ 0 & 0 & 1 & 38 & -13 & -22 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -79 & 27 & 46 \\ 0 & 1 & 0 & 12 & -4 & -7 \\ 0 & 0 & 1 & 38 & -13 & -22 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix}$

$x = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}$

27. An $n \times n$ matrix A is called **orthogonal** if $A^T = A^{-1}$. For Problems 27–30, show that the given matrices are orthogonal.

27. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

28. $A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$

29. $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$

27. $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$

$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

29. $A^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$\left[\begin{array}{cc|cc} \cos\alpha & \sin\alpha & 1 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \tan\alpha & \sec\alpha & 0 \\ -1 & \cot\alpha & 0 & \csc\alpha \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & \tan\alpha & \sec\alpha & 0 \\ 0 & \tan\alpha + \cot\alpha & \sec\alpha & \csc\alpha \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & \tan\alpha & \sec\alpha & 0 \\ 0 & \frac{1}{\sin\alpha\cos\alpha} & \frac{1}{\cos\alpha} & \frac{1}{\sin\alpha} \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & \tan\alpha & \sec\alpha & 0 \\ 0 & 1 & \sin\alpha & \cos\alpha \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & \tan\alpha & \cos\alpha & -\sin\alpha \\ 0 & 1 & \sin\alpha & \cos\alpha \end{array}\right]$

2.7 - 3, 10, 14, 18, 22, 28

For Problems 2–6, determine elementary matrices that reduce the given matrix to row-echelon form.

2. $\begin{bmatrix} -4 & -1 \\ 0 & 3 \\ -3 & 7 \end{bmatrix}$.

3. $\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}$.

4. $\begin{bmatrix} 5 & 8 & 2 \\ 1 & 3 & -1 \end{bmatrix}$.

5. $\begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$.

3. $\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

① P_{12} ② $A_{12}(-3)$ ③ $M_2(\frac{1}{11})$

$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

For Problems 7–13, express the matrix A as a product of elementary matrices.

7. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

8. $A = \begin{bmatrix} -2 & -3 \\ 5 & 7 \end{bmatrix}$.

9. $A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$.

10. $A = \begin{bmatrix} 4 & -5 \\ 1 & 4 \end{bmatrix}$.

11. $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$.

12. $A = \begin{bmatrix} 0 & -4 & -2 \\ 1 & -1 & 3 \\ -2 & 2 & 2 \end{bmatrix}$.

13. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 0 \\ 3 & 4 & 5 \end{bmatrix}$.

10. $\begin{bmatrix} 4 & -5 \\ 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & -21 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

① P_{12} ② $A_{12}(-4)$ ③ $M_2(-\frac{1}{21})$ ④ $A_{21}(-4)$

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{21} \end{bmatrix}$ $E_4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -21 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

14. Determine elementary matrices $E_1, E_2, \ldots, E_k$ that reduce

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

to reduced row-echelon form. Verify by direct multiplication that $E_1 E_2 \ldots E_k A = I_2$.

14. $A \sim \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

① P_{12} ② $A_{12}(-2)$ ③ $M_2(-\frac{1}{7})$ ④ $A_{21}(-3)$

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}$ $E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3/7 \\ 0 & -\frac{1}{7} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3/7 & 1/7 \\ -1/7 & 2/7 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

For Problems 16–21, determine the LU factorization of the given matrix. Verify your answer by computing the product LU.

16. $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$.

17. $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

18. $A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix}$.

19. $A = \begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix}$.

20. $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix}$.

21. $A = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix}$.

18. $A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ -3 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 4 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 16 \end{bmatrix} = U$

① $A_{12}(-2)$, $A_{13}(1)$, $A_{23}(-4) \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$

$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 16 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix}$

For Problems 22–25, use the LU factorization of A to solve the system $A\mathbf{x} = \mathbf{b}$.

22. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

23. $A = \begin{bmatrix} 1 & -3 & 5 \\ 3 & 2 & 2 \\ 2 & 5 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$.

24. $A = \begin{bmatrix} 2 & 2 & 1 \\ 6 & 3 & -1 \\ -4 & 2 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

22. $Ly = b \Rightarrow Ux = y$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = U$

① $A_{12}(-2)$ $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & 1 & -1 \end{array}\right]$: $y = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & -7 \end{array}\right] = x = \begin{bmatrix} -11 \\ 7 \end{bmatrix}$

$x_1 = -11$, $x_2 = 7$

28. If $P = P_1 P_2 \ldots P_k$, where each P_i is an elementary permutation matrix, show that $P^{-1} = P^T$.

28. Permutation: $P_i = P_i^{-1} = P_i^T$

$P^{-1} = P_k^{-1} P_{k-1}^{-1} \cdots P_1^{-1} = P_k^T P_{k-1}^T \cdots P_1^T$

$= (P_1 P_2 \cdots P_k)^T = P^T$