MATH 225 Homework 3

Stanley Hong

February 18, 2022

Problem 1: Goode 3.1.2

Determine the number of inversions and the parity for the given permutation: $(2, 4, 3, 1)$.

Solution. $(2, 4, 3, 1) \to (2, 4, 1, 3) \to (2, 1, 4, 3) \to (1, 2, 4, 3) \to (1, 2, 3, 4)$. Number of inversions = 4, even parity.

Problem 2: Goode 3.1.5

Determine the number of inversions and the parity for the given permutation: $(6, 1, 4, 2, 5, 3)$.

Solution. $(6, 1, 4, 2, 5, 3) \to (1, 6, 4, 2, 5, 3) \to (1, 2, 6, 4, 5, 3) \to (1, 2, 4, 6, 5, 3) \to (1, 2, 4, 6, 3, 5) \to (1, 2, 4, 3, 6, 5) \to (1, 2, 3, 4, 6, 5) \to (1, 2, 3, 4, 5, 6)$. Number of inversions = 8, even parity.

Problem 3: Goode 3.1.17

Evaluate the determinant of the matrix $A = \begin{bmatrix} 6 & -3 \\ -5 & -1 \end{bmatrix}$.

Solution. $\det(A) = ad - bc = -6 - 15 = -21$.

Problem 4: Goode 3.1.27

Evaluate the determinant of the matrix $A = \begin{bmatrix} -2 & -4 & 1 \\ 6 & 1 & 1 \\ -2 & -1 & 3 \end{bmatrix}$.

Solution. $\det(A) = -2 \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} - (-4) \begin{vmatrix} 6 & 1 \\ -2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 6 & 1 \\ -2 & -1 \end{vmatrix} = -8 + 80 - 4 = 68.$

Problem 5: Goode 3.1.39

Evaluate the determinant of the matrix $A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -8 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}$.

Solution. $\det(A) = -1\begin{vmatrix} -8 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -1 & -1 \end{vmatrix} = (-1)(-8)(-2+3) - (2)(2)(-2+3) = 4.$

Problem 6: Goode 3.1.45

Evaluate the determinant of the matrix function $A = \begin{bmatrix} e^{2t} & e^{3t} & e^{-4t} \\ 2e^{2t} & 3e^{3t} & -4e^{-4t} \\ 4e^{2t} & 9e^{3t} & 16e^{-4t} \end{bmatrix}$.

Solution. $\det(A) = e^{2t}\begin{vmatrix} 3e^{3t} & -4e^{-4t} \\ 9e^{3t} & 16e^{-4t} \end{vmatrix} - e^{3t}\begin{vmatrix} 2e^{2t} & -4e^{-4t} \\ 4e^{2t} & 16e^{-4t} \end{vmatrix} + e^{-4t}\begin{vmatrix} 2e^{2t} & 3e^{3t} \\ 4e^{2t} & 9e^{3t} \end{vmatrix}$
$= e^{2t}(84e^{-t}) - e^{3t}(48e^{-2t}) + e^{4t}(6e^{5t}) = 42e^{t}.$

Problem 7: Goode 3.2.5

Evaluate the determinant of the matrix $A = \begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$ by reducing it to upper triangular form.

Solution. We use theorem 3.2.1. If A is an $n \times n$ upper or lower triangular matrix, then $\det(A) = \prod_{i=1}^{n} a_{ii}$. This way, $A = \begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 2 & -3 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{vmatrix}$, $\det(A) = (1)(1)(0) = 0$.

Problem 8: Goode 3.2.11

Evaluate the determinant of the matrix $A = \begin{vmatrix} 2 & -1 & 3 & 4 \\ 7 & 1 & 2 & 3 \\ -2 & 4 & 8 & 6 \\ 6 & -6 & 18 & -24 \end{vmatrix}$ by reducing it to upper triangular form.

Solution. $A \sim \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 4.5 & -8.5 & -11 \\ 0 & 3 & 11 & 10 \\ 0 & -3 & 9 & -36 \end{vmatrix} \sim \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 4.5 & -8.5 & 11 \\ 0 & 0 & \frac{50}{3} & \frac{52}{3} \\ 0 & 0 & 20 & -26 \end{vmatrix} \sim \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 4.5 & -8.5 & 11 \\ 0 & 0 & \frac{50}{3} & \frac{52}{3} \\ 0 & 0 & 0 & -\frac{234}{5} \end{vmatrix}$, $\det(A) = -7020$.

Problem 9: Goode 3.2.18

Use theorem 3.2.5 (A is invertible $\Leftrightarrow$ $\det A \neq 0$) to determine whether $A = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 5 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ is invertible or not.

Solution. We use theorem 3.2.5 by calculating the determinant. $\det(A) = (2)(5) - (6)(1) + (-1)(-10) = 14 \neq 0$, so A is invertable.

Problem 10: Goode 3.2.22

Determine all values of the constant k for which the given system has an infinite number of solutions.

$$\begin{cases} x_1 + 2x_2 + kx_3 = 0 \\ 2x_1 - kx_2 + x_3 = 0 \\ 3x_1 + 6x_2 + x_3 = 0 \end{cases}$$

Solution. By corollary 3.2.6, the homogeneous $n \times n$ linear system has an infinite number of solutions if and only if $\det(A) = 0$. Computing the determinant for $A = \begin{bmatrix} 1 & 2 & k \\ 2 & -k & 1 \\ 3 & 6 & 1 \end{bmatrix}$, we have

$\det(A) = (1)(-k-6) - 2(2-3) + k(12+3k) = -4 + 11k + 3k^2 = (3k-1)(k+4) = 0$, so $k = \left\{\frac{1}{3}, -4\right\}$.

Problem 11: Goode 3.2.26

If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$, find $\det(A)$, and use properties of determinants to find $\det(A^{-1})$ and $\det(-3A)$.

Solution. $\det(A) = (1)(3-4) - (-1)(9) + 2(3) = -1 + 9 + 6 = 14$. $\det(A^{-1}) = \frac{1}{14}$, $\det(-3A) = (-3)^3(14) = -378$.

Problem 12: Goode 3.2.33

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and assume $\det(A) = -6$. Find $\det(B)$ if $B = \begin{bmatrix} g & h & i \\ -2d & -2e & -2f \\ -a & -b & -c \end{bmatrix}$.

Solution. We can see that B can be represented as A after row permutation and multiplication operations. Hence $\det(B) = \det(A)(-2)(-1) = 12$.

Problem 13: Goode 3.2.46

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$.

(a) In terms of k, find the volume of the parallelepiped determined by the row vectors of the matrix A.

(b) Does your answer to (a) change if we instead consider the volume of the parallelepiped determined by the column vectors of the matrix A? Why or why not?

(c) For what value(s) of k, if any, is A invertible?

Solution. (a) $\det(A) = 1(2-18) - 2(6-6k) + 4(9-k) = -16 - 12 + 12k + 36 - 4k = 8k + 8$. The volume of the parallelepiped will then by $V = |\det(A)| = |8 + 8k|$.

(b) The answer does not change. The new matrix formed by the column vectors will be the transpose of the original vector, and $\det(A) = \det(A^T)$.

(c) A is invertible when the determinant is not zero. Hence, if $k \neq -1$, A is invertible.

Problem 14: Goode 3.2.58

Without expanding the determinant, show that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-z)(z-x)(x-y)$.

Solution. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \sim \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \sim (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} \sim (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix}$.

Then, $\det(A) = (y-x)(z-x)(z-y) = (y-z)(z-x)(x-y)$.

Problem 15: Goode 3.3.3

Determine all minors and cofactors of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 5 \end{bmatrix}$.

Solution. Minors: $M_{11} = -9$, $M_{12} = 7$, $M_{13} = 5$, $M_{21} = -7$, $M_{22} = 1$, $M_{23} = 3$, $M_{31} = -2$, $M_{32} = -2$, $M_{33} = 2$.

Cofactors: $C_{11} = -9$, $C_{12} = -7$, $C_{13} = 5$, $C_{21} = 7$, $C_{22} = 1$, $C_{23} = -3$, $C_{31} = -2$, $C_{32} = 2$, $C_{33} = 2$.

Problem 16: Goode 3.3.11

Use the Cofactor Expansion Theorem to evaluate $\begin{vmatrix} 3 & 1 & 4 \\ 7 & 1 & 2 \\ 2 & 3 & -5 \end{vmatrix}$ along column 1.

Solution. The Cofactor Expansion Theorem tells us that $\det(A) = \sum_{k=1}^{n} a_{kj} C_{kj}$.

Then, $\begin{vmatrix} 3 & 1 & 4 \\ 7 & 1 & 2 \\ 2 & 3 & -5 \end{vmatrix} = 3(-11) - 7(-17) + 2(-2) = -33 + 119 - 4 = 82$.

Problem 17: Goode 3.3.16

Evaluate $\begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 7 & 2 & 5 \end{vmatrix}$ using the Cofactor Expansion Theorem. Do not apply elementary row operations.

Solution. $\det(A) = (1)(7) + (-2)(-1) = 9$.

Problem 18: Goode 3.3.23

Use elementary row operations together with the Cofactor Expansion Theorem to evaluate $\begin{vmatrix} -1 & 3 & 3 \\ 4 & -6 & 3 \\ 2 & -1 & 4 \end{vmatrix}$.

Solution. $\begin{vmatrix} -1 & 3 & 3 \\ 4 & -6 & 3 \\ 2 & -1 & 4 \end{vmatrix} \sim \begin{vmatrix} -1 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 5 & 10 \end{vmatrix} \sim 6 \begin{vmatrix} -1 & 3 & 3 \\ 0 & 1 & 1.5 \\ 0 & 0 & 2.5 \end{vmatrix}$. Now$\det(A) = (6)(-1)(2.5) = -15$.

Problem 19: Goode 3.3.30

(a) Consider the 3×3 *Vandermonde* determinant $V(r_1, r_2, r_3)$ defined by $V(r_1, r_2, r_3) = \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^3 \end{vmatrix}$. Show that $V(r_1, r_2, r_3) = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2)$.

(b) More generally, show that the $n \times n$ Vandermonde determinant $V(r_1, r_2, \dots, r_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{r-1} & \dots & r_n^{n-1} \end{vmatrix}$ has value $V(r_1, r_2, \dots, r_n) = \prod_{1 \leqslant i < m \leqslant n} (r_m - r_i)$.

Solution. (a) Refer to problem 14: Goode 3.2.58. The matrix is the transpose of the one in 3.2.58, so it has the same determinant.

(b) Because $\det(A) = \det(A^T)$, we consider its transpose $V^T = \begin{vmatrix} 1 & r_1 & \dots & r_1^{n-1} \\ 1 & r_2 & \dots & r_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & r_n & \dots & r_n^{n-1} \end{vmatrix}$. If we do $A_{1j}(-1)$ for $j \geqslant 2$, then we have $V^T \sim \begin{vmatrix} 1 & r_1 & \dots & r_1^{n-1} \\ 0 & r_2 - r_1 & \dots & r_2^{n-1} - r_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & r_n - r_1 & \dots & r_n^{n-1} - r_1^{n-1} \end{vmatrix}$. We can then divide $r_j - r_1$ from all $j \geqslant 2$, which gives $V^T \sim (r_2 - r_1) \times \dots \times (r_n - r_1) \begin{vmatrix} 1 & r_1 & \dots & r_1^{n-1} \\ 0 & 1 & \dots & \frac{r_2^{n-1} - r_1^{n-1}}{r_2 - r_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & \frac{r_n^{n-1} - r_1^{n-1}}{r_n - r_1} \end{vmatrix}$. We apply the same technique by $A_{2j}(-1)$ for $j \geqslant 3$, then the new matrix will then be multiplied with $(r_3 - r_2) \times \dots \times (r_n - r_2)$. Repeat the process and we obtain $V^T \sim \prod_{1 \leqslant i < j \leqslant n} (r_j - r_i) \left| A \right|$, where A is an upper triangular matrix and $A_{ii} = 1$. Then we obtain the answer.

Problem 20: Goode 3.3.33

Determine the eigenvalues of the matrix $A = \begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix}$. i.e., determine the scalars λ such that $\det(A - \lambda I) = 0$.

Solution. $A - \lambda I = \begin{bmatrix} -1 - \lambda & 2 \\ -4 & 7 - \lambda \end{bmatrix}$ We want to make $\det(A - \lambda I) = 0$, so we obtain the quadratic equation $\lambda^2 - 6\lambda - 7 + 8 = 0$. Solving the equation gives $\lambda_1 = 2\sqrt{2} + 3$, $\lambda_2 = -2\sqrt{2} + 3$.

Problem 21: Goode 3.3.43

For $A = \begin{bmatrix} -2 & 3 & -1 \\ 2 & 1 & 5 \\ 0 & 2 & 3 \end{bmatrix}$, find (a) $\det(A)$, (b) the matrix of cofactors M_C, (c) $\operatorname{adj}(A)$, and, if possible, (d) A^{-1}.

Solution. (a) $\det(A) = (-2)(-7) - (3)(6) + (-1)(4) = -8$. (b) $M_C = \begin{bmatrix} -7 & -6 & 4 \\ -11 & -6 & 4 \\ 16 & 8 & -8 \end{bmatrix}$.

(c)$\operatorname{adj}(A) = M_C^T = \begin{bmatrix} -7 & -11 & 16 \\ -6 & -6 & 8 \\ 4 & 4 & -8 \end{bmatrix}$. (d) $A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = -\frac{1}{8} \begin{bmatrix} -7 & -11 & 16 \\ -6 & -6 & 8 \\ 4 & 4 & -8 \end{bmatrix}$.

Problem 22: Goode 3.3.59

Use Cramer's rule to solve the linear system:

$$\begin{cases} 2x_1 - 3x_2 = 2 \\ x_1 + 2x_2 = 4 \end{cases}$$

Solution. First we know $\det(A) = 7 \neq 0$, so the solution is unique. Now we replace $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ with the k-th column of A, which gives us $B_1 = \begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$. Then following Cramer's rule, we obtain $x_1 = \frac{16}{7}$, $x_2 = \frac{6}{7}$.

Problem 23: Goode 3.3.63

Use Cramer's rule to solve the linear system:

$$\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ x_1 + 4x_2 - x_3 = 0 \\ 2x_1 + x_2 - 3x_3 = 0 \end{cases}$$

Solution. First we know that $\det(A) = -11 - 3 - 7 = -21$. Since $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then there only exists one trivial solution to the homogeneous linear system. Hence, we obtain $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.