ECON 577 Homework 1

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Problem. Show that for constant relative risk averse (CRRA) utility

$$u(W) = \frac{W^{1-\gamma}}{1-\gamma},$$

the function converges to $u(W) = \ln(W)$ when $\gamma \to 1$. Compute the coefficient of absolute risk aversion and relative risk aversion for $\log$ utility.

Sol.

Proof of convergence. Consider the substitution $h := 1 - \gamma$. We have that

$$\lim_{\gamma\to 1} \frac{W^{1-\gamma}}{1-\gamma} = \lim_{h\to 0} \frac{W^h}{h} \overset{\text{L'Hôpital}}{=} \lim_{h\to 0} W^h \ln W = \ln W.$$

Direct computation gives

$$(\ln W)' = W^{-1}, \quad (W^{-1})' = -W^{-2}.$$

Therefore

$$A(W) = W^{-1}, \quad R(W) = 1.$$

the $\log$ utility function indeed exhibits constant relative risk!

Problem. Assume constant absolute risk averse (CARA) utility. Assume that wealth is a normally distributed random variable with $W \sim \mathrm{Normal}(\mu, \sigma^2)$. Compute the certainty equivalent $c^e(\mu, \sigma^2, A, W_0)$, where W_0 is the investor's initial wealth. Explain the intuition.

Sol. Considering the risk, u follows a $\log$-normal distribution as the underlying normal variable can be written as $-Aw \sim \mathrm{Normal}(-A\mu, A^2\sigma^2)$ and hence utility expectation

$$\mathbb{E}(u(W)) = -\frac{1}{A} \exp\left(-A\mu + \frac{A^2\sigma_W^2}{2}\right).$$

Considering the definition of certainty equivalent, we have that

$$u(W_0 + c^e) = -\frac{1}{A} \exp\left(-AW_0 - Ac^e\right) \overset{?}{=} -\frac{1}{A} \exp\left(-A\mu + \frac{A^2\sigma_W^2}{2}\right).$$

Computation gives

$$c^e = \mu - W_0 - \frac{A\sigma_W^2}{2}.$$

Note that the certainty equivalent c^e is linear in initial wealth W_0; in fact, $W_0 + c^e$ is *constant*, implying that any agent would want a certainty equivalent so that its wealth W^* (assuming $W^* > W_0$, or else the certainty equivalent would be zero) equates

$$W^* = \mu - \frac{A\sigma_W^2}{2}.$$

Problem. Derive the Arrow-Pratt risk premium assuming that $\tilde{y}$ is not zero mean. Specifically, assume that $\tilde{y} \sim \text{Normal}(\mu, \sigma^2)$.

Proof. Now that the risk takes two variables $\tilde{x} \sim \text{Normal}(\mu, \sigma^2)$. Therefore it makes more sense to consider a *two-dimensional Taylor expansion* as follows:

$$\pi(\mu, \sigma) \approx \pi(0,0) + \mu\pi_\mu(0,0) + \sigma\pi_\sigma(0,0) + \frac{1}{2}\left[\pi_{\mu\mu}(0,0)\mu^2 + 2\pi_{\mu\sigma}(0,0)\mu\sigma + \pi_{\sigma\sigma}(0,0)\sigma^2\right].$$

We first consider the risk premium:

$$\mathbb{E}\left[u(W_0 + \sigma\tilde{x} + \mu)\right] = u(W_0 - g(\sigma, \mu)).$$

$\boldsymbol{g_\sigma}$ **and** $\boldsymbol{g_{\sigma\sigma}}$**.** We differentiate the equation with respect to σ first.

$$\mathbb{E}\left[\tilde{x}u_\sigma(W_0 + \sigma\tilde{x} + \mu)\right] = -g_\sigma(\sigma, \mu)u_\sigma(W_0 - g(\sigma, \mu)).$$

The left-hand side equates $\mathbb{E}\left[\tilde{x}\right]u_\sigma(W_0 + \sigma\tilde{x} + \mu) = 0$, and therefore $g_\sigma(\sigma, \mu) = 0$.
Differentiating with respect to σ again, we have that

$$\mathbb{E}\left[\tilde{x}^2 u_{\sigma\sigma}(W_0 + k\tilde{x} + \mu)\right] = -g_{\sigma\sigma}(\sigma, \mu)u_\sigma(W_0 - g(\sigma, \mu)) + u_{\sigma\sigma}(W_0 - g(\sigma, \mu))g_\sigma^2(\sigma, \mu).$$

As $g_\sigma = 0$, the second term is omitted. We then can obtain

$$g_{\sigma\sigma}(\sigma, \mu) = -\frac{\mathbb{E}\left[\tilde{x}^2\right]u_{\sigma\sigma}(W_0 + \sigma\tilde{x} + \mu)}{u_\sigma(W_0 - g(\sigma, \mu))}.$$

Considering $\sigma, \mu = 0$, the expression simplifies to

$$g_{\sigma\sigma}(\sigma, \mu) = -\mathbb{E}\left[\tilde{x}^2\right]A(W_0).$$

$\boldsymbol{g_\mu}$ **and** $\boldsymbol{g_{\mu\mu}}$**.** We then differentiate with respect to μ.

$$\mathbb{E}\left[u_\mu(W_0 + \sigma\tilde{x} + \mu)\right] = -g_\mu(\sigma, \mu)u_\mu(W_0 - g(\sigma, \mu)).$$

We then can simplify the expression to obtain

$$g_\mu(\sigma, \mu) = -\frac{u_\mu(W_0 + \sigma\tilde{x} + \mu)}{u_\mu(W_0 - g(\sigma, \mu))} \Rightarrow g_\mu(0,0) = -1.$$

Differentiating with respect to μ a second time returns

$$\mathbb{E}\left[u_{\mu\mu}(W_0 + \sigma\tilde{x} + \mu)\right] = -g_{\mu\mu}(\sigma, \mu)u_\sigma(W_0 - g(\sigma, \mu)) + u_{\mu\mu}(W_0 - g(k, \mu))g_\mu^2(\sigma, \mu).$$

Evaluated at $(0,0)$, we see that $g_\mu^2(0,0) = 1$, so that the $u_{\mu\mu}$ terms cancel out to zero. Thus $g_{\mu\mu}(\sigma, \mu) = 0$.
$\boldsymbol{g_{\sigma\mu}}$**.** Lastly we consider the $g_{\sigma\mu}$ derivative. Differentiation gives

$$\mathbb{E}\left[\tilde{x}u_{\sigma\mu}(W_0 + \sigma\tilde{x} + \mu)\right] = -g_{\sigma\mu}(\sigma, \mu)u_\sigma(W_0 - g(\sigma, \mu)) - u_{\sigma\mu}(W_0 - g(\sigma, \mu))g_\sigma(\sigma, \mu)g_\mu(\sigma, \mu).$$

The left-hand side equals zero as $\mathbb{E}[\tilde{x}] = 0$; the second term of the right-hand side also equals zero as the first derivative $g_\sigma(\sigma,\mu) = 0$. Therefore we see that the term $g_{\sigma\mu}(\sigma,\mu)u_\sigma(W_0 - g(\sigma,\mu)) = 0 \Rightarrow g_{\sigma\mu}(\sigma,\mu) = 0$ as well.

Series expansion. Lastly we could expand the Taylor series from our previous findings.

$$\pi(\mu,\sigma) \approx -\mu + \frac{1}{2}A\sigma^2\mathbb{E}\left[\tilde{x}^2\right].$$

A direct interpretation to the expression is that although risk premium is still quadratic in the risk $\tilde{x}$, it is, in fact, linear to the mean μ, decreasing as μ increases. □