ECON 577 Homework 7

Stanley Hong

Due November 8, 2023

Remark. Throughout the assignment, I used $\mathbb{Q}$ to represent the risk-neutral measure. Albeit different from the textbook representation, I believe they convey the same idea.

Problem: Shreve 2.2. Consider the stock price S_3 in figure 2.3.1. (This is the same multiperiod binomial model we have been discussing.)

(a) What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$?

(b) Compute $\mathbb{E}^{\mathbb{Q}}[S_1]$, $\mathbb{E}^{\mathbb{Q}}[S_2]$, and $\mathbb{E}^{\mathbb{Q}}[S_3]$. What is the average rate of growth of the stock price under $\mathbb{Q}$?

(c) Answer (a) and (b) again under the actual probabilities $p = \frac{2}{3}$, $q = \frac{1}{3}$.

Sol of parts (a)(b). $P(X_3 = 32) = P(X_3 = 0.5) = 0.125$, $P(X_3 = 8) = P(X_3 = 2) = 0.375$. Thus,

$$\mathbb{E}^{\mathbb{Q}}[S_1] = \frac{1}{2}S_1(H) + \frac{1}{2}S_1(T) = 5, \quad \mathbb{E}^{\mathbb{Q}}[S_2] = \frac{25}{4}, \quad \mathbb{E}^{\mathbb{Q}}[S_3] = \frac{125}{16}.$$

The average rate of growth is $5/4 - 1 = 25\%$ over a period.

Sol of part (c). $P(X_3 = 32) = \frac{8}{27}$, $P(X_3 = 8) = \frac{12}{27}$, $P(X_3 = 2) = \frac{8}{27}$, $P(X_3 = 0.5) = \frac{1}{27}$. (The numbers were not simplified for convenience in future calculations.) Therefore,

$$\mathbb{E}[S_1] = \frac{2}{3}S_1(H) + \frac{1}{3}S_1(T) = 6, \quad \mathbb{E}[S_2] = 9, \quad \mathbb{E}[S_3] = 13.5.$$

The average rate of growth is $3/2 - 1 = 50\%$ over a period.

Problem: Shreve 2.4. Toss a coin repeatedly. Assume the probability of head on each toss is $\frac{1}{2}$, as is the probability of tail. Let $X_j = 1$ if the j-th toss results in a head, and $X_j = -1$ if the j-th toss results in a tail. Consider the stochastic process $M_0, M_1, M_2, \cdots$ defined by $M_0 = 0$ and

$$M_n = \sum_{k=1}^{n} X_j, \quad n \geqslant 1,$$

(This is called a *symmetric random walk.*)

(a) Using the properties of theorem 2.3.2, show that $M_0, M_1, \cdots$ is a martingale.

(b) Let σ be a positive constant and for $n \geqslant 0$ define

$$S_n = e^{\sigma M_n} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n .$$

Show that $S_0, S_1, \cdots$ is a martingale. Note that even through M_n has no tendency to grow, the *geometric symmetric random walk* $e^{\sigma M_n}$ has a tendency to grow as a result of putting a martingale into the (convex) exponential function. We add the coefficient to "discount" the geometric symmetric random walk, with the coefficient $2(e^{\sigma} + e^{-\sigma})^{-1} < 1$ unless $\sigma = 0$.

Sol of part (a). It suffices to show that $M_n = \mathbb{E}_n [M_{n+1}]$, and the proof follows by recursion. Specifically, we have that

$$\begin{aligned} \mathbb{E}_n [M_{n+1}] &= \mathbb{E}_n [M_n] + \mathbb{E}_n [X_{n+1}] \\ &= M_n + 0 = M_n, \end{aligned}$$

which proves the statement.

Sol of part (b). Again, we wish to show $S_n \overset{?}{=} \mathbb{E}_n [S_{n+1}]$. We see that

$$\begin{aligned} \mathbb{E}_n [S_{n+1}] &= \frac{1}{2} \exp(\sigma(M_n + 1)) \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} + \frac{1}{2} \exp(\sigma(M_n - 1)) \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} \\ &= \left[\frac{1}{2} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} \cdot e^{\sigma M_n} \right] (e^{\sigma} + e^{-\sigma}) \\ &= \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n e^{\sigma M_n} = S_n, \end{aligned}$$

as desired.

Problem: Shreve 2.8. Consider an N-period binomial model.

(a) Let $M_0, \cdots, M_n$ and $M'_0, \cdots, M'_N$ be martingales under the risk-neutral measure $\mathbb{Q}$. Show that if $M_N = M'_N$ for every possible outcome of the sequence of coin tosses, then, for each n between 0 and N, we have $M_n = M'_n$ for every possible outcome of the sequence of coin tosses.

(b) Let V_N be the payoff at time N of some derivative security. This is a random variable that can depend on all N coin tosses. Define recursively $V_{N-1}, \cdots, V_0$ by the algorithm (1.2.16) (discounted payoff algorithm) in chapter 1. Show that

$$V_0, \frac{V_1}{1+r}, \cdots, \frac{V_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale under $\mathbb{Q}$.

(c) Using the risk-neutral pricing formula, define

$$V'_n = \mathbb{E}^{\mathbb{Q}}_n \left[\frac{V_N}{(1+r)^{N-n}} \right], n \in [N-1]_0 .$$

Show that

$$V'_0, \frac{V'_1}{1+r}, \cdots, \frac{V'_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale.

(d) Conclude that $V_n = V'_n$ for every n, i.e., the algorithm 1.2.16 gives the same derivative security prices as the risk-neutral pricing formula.

Sol of part (a). Note that $M_n = \mathbb{E}_n[M_N]$ and $M'_n = \mathbb{E}_n[M'_N]$. As $M_N = M'_N$ for every outcome, they have the same expectations and therefore

$$M_n = M'_n$$

for every possible outcome as well.

Sol of part (b). V_N can be represented as the value process of a portfolio of stocks and bonds. Namely, it can be achieved by $V_0 = X_0$. And theorem 2.4.5 states that the discounted wealth process $(\frac{X_n}{(1+r)^n})_n$ is a martingale under the risk-neutral measure $\mathbb{Q}$. Thus the discounted value in $(\frac{V_n}{(1+r)^n})_n$ is also a martingale under the same measure.

Sol of part (c). We wish to show that

$$\mathbb{E}\left[\frac{V'_n}{(1+r)^n} \right] = V'_0.$$

The case where $n = N$ is obvious. As for the general case, we see that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}_0 \left[\frac{V'_n}{(1+r)^n} \right] &= \mathbb{E}^{\mathbb{Q}}_0 \left[\frac{\mathbb{E}^{\mathbb{Q}}_n \left[\frac{V_N}{(1+r)^{N-n}} \right]}{(1+r)^n} \right] \\ &= \mathbb{E}^{\mathbb{Q}}_0 \left[\mathbb{E}^{\mathbb{Q}}_n [V_N] (1+r)^{-N} \right] = V'_0. \end{aligned}$$

Sol of part (d). Given that the process in (b) is a martingale, the process in (c) is a martingale, and that $V_N = V'_N$ (as they are defined the same), by (a) it follows that $V_n = V'_n$.