Chapter 2 - Matrices

Square matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Main diagonal: $a_{11}, a_{22}, \dots, a_{nn}$; **trace**: $\sum_{i=1}^{n} a_{ii}$

Symmetric: $A^T = A$; **skew-symmetric**: $A^T = -A$

Linear equation system $Ax = b$ (**homogeneous** if $b = 0$)

Row-echelon matrix: (1) all zero rows at the bottom, (2) all other rows begin with leading "1", (3) leading "1"'s occur strictly to the right of the leading "1"'s above

Rank: no. of nonzero rows in row-echelon form

Invertible matrix: $AA^{-1} = A^{-1}A = I_n$

$$\left[A \mid I_n\right] \sim \dots \sim \left[I_n \mid A^{-1}\right]$$

$$(A^{-1})^{-1} = A \;\; (AB)^{-1} = B^{-1}A^{-1} \;\; (A^T)^{-1} = (A^{-1})^T$$

Elementary matrix:

$$P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;\; M_1(k) = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \;\; A_{12}(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

$$M_i(k)^{-1} = M_i(k^{-1}) \;\; P_{ij}^{-1} = P_{ij} \;\; Aij(k)^{-1} = A_{ij}(-k)$$

LU factorization:

$$E_k E_{k-1} \dots E_2 E_1 A = U \Rightarrow A = E_1^{-1} E_2^{-1} \dots E_k^{-1} U = LU$$

Chapter 3 - Determinants

Triangular matrix determinant:

$$\text{det}(A) = \prod_{i=1}^{n} a_{ii}$$

Matrix determinant rules: Interchanging rows $\Rightarrow$ $-\text{det}(A)$, multiplying by scalar $\Rightarrow$ $k\text{det}(A)$, adding rows $\Rightarrow$ $\text{det}(A)$, $\text{det}(AB) = \text{det}(A)\text{det}(B)$, $\text{det}(A^{-1}) = (\text{det}(A))^{-1}$

Cofactor: $C_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} is the determinant obtained from deleting row i and col j

Cofactor expansion theorem:

$$\text{det}(A) = \sum_{k=1}^{n} a_{ij} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj}$$

Adjoint method of inverse:

$$A^{-1} = (\text{det}(A))^{-1} \text{adj}(A), \;\; \text{adj}(A)_{ij} = C_{ji}$$

Cramer's rule: solution to $Ax = b$ is $(x_1, \dots, x_n)$, where

$$x_k = \frac{\text{det}(B_k)}{\text{det}(A)}, \;\; B_k = \begin{bmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{bmatrix}$$

Chapter 4 - Vector spaces

Vector space: closure under "+" and "×"

$$\forall u, v \in V, \; u + v \in V; \;\; \forall k \in \mathbb{R}, \forall v \in V, \; kv \in V$$

Subspace: $S \neq \varnothing$, $S \subset V$, S vector space under "+" and "×"

Nullspace: $\text{nullspace}(A) = \{x : Ax = 0\}$

Spanning set: $\{v_1, \dots, v_k\}$ **spans** V if

$$\forall v \in V, v = c_1 v_1 + \dots + c_k v_k$$

Linear dependency: $\{v_1, \dots, v_k\}$ linearly dependent if

$$c_1 v_1 + \dots + c_k v_k = 0$$

for some nonzero $(c_1, \dots, c_k)$

If $W\left[f_1, \dots, f_k\right] \neq 0$ at some $x_0 \in I$, then $\{f_1, \dots, f_k\}$ linearly independent on I. Namely,

$$W = \begin{bmatrix} f_1(x) & \dots & f_k(x) \\ \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & \dots & f_k^{(k-1)}(x) \end{bmatrix}$$

Basis: $\{v_1, \dots, v_k\}$ span V, and are linearly independent

Dimension: no. of vectors in any basis for V

$\dim\left[V\right] = n \Rightarrow$ set of $> n$ vectors linearly dependent, set of n linearly independent vectors in V is a basis for V

Component rel. to ordered basis $B = \{v_1, \dots, v_k\}$:

$$\left[v\right]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \Rightarrow c = c_1 v_1 + \dots + c_k v_k$$

Change of basis matrix $B = \{v_1, \dots, v_k\} \to \{w_1, \dots, w_k\}$:

$$P_{C \leftarrow B} = \left[\left[v_1\right]_C \quad \dots \quad \left[v_n\right]_C\right]$$

$$\left[v\right]_C = P_{C \leftarrow B} \left[v\right]_B$$

Row space: row vectors of A that span a subspace of $\mathbb{R}^n$

Column space: col vectors of A that span a subspace of $\mathbb{R}^m$

Set of col vectors of A corresponding to col vectors containing leading ones in row-echelon form of A is a basis for colspace

Rank-nullity theorem:

$$\text{rank}(A) + \text{nullity}(A) = n$$

Invertible matrix theorem: A invertible $\equiv$ A^T invertible $\equiv$ $Ax = b$ unique solution $\equiv$ $Ax = 0$ trivial solution $\equiv$ $\text{rank}(A) = n$ $\equiv$ $\text{nullity}(A) = 0$ $\equiv$ $\text{nullspace}(A) = \{0\}$ $\equiv$ $\text{colspace}(A) = \mathbb{R}^n$ $\equiv$ $\text{rowspace}(A) = \mathbb{R}^n$ $\equiv$ cols/rows of A form a basis for $\mathbb{R}^n$

Chapter 6 - Linear transformation

Linear transformation: mapping with linearity properties

$$\forall u, v \in V,\ T(u+v) = T(u) + T(v)$$

$$\forall k \in \mathbb{R}, \forall v \in V,\ T(cv) = kT(v)$$

Matrix of transformation: $m \times n$ matrix corresponding to the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$: $T(x) = Ax$

$$A = \begin{bmatrix} T(e_1) & \dots & T(e_n) \end{bmatrix}$$

Transformation of $\mathbb{R}^2$: reflection, shear, stretch

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\ R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\ R_{xy} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$LS_x = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\ LS_y = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\ S_x = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\ S_y = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Transformation with invertible matrix:

$$T(v) = Av = E_1^{-1} \dots E_n^{-1} v$$

Kernel: $\text{Ker}(T) = \{v \in V : T(v) = 0\}$
Range: $\text{Rng}(T) = \{T(v) : v \in V\}$
For $T : \mathbb{R}^n \to \mathbb{R}^m$, with $T(v) = Av$,

$$\text{Ker}(T) = \text{nullspace}(A) \subset \mathbb{R}^n \quad \text{Rng}(T) = \text{colspace}(A) \subset \mathbb{R}^m$$

General rank-nullity theorem: for $T : V \to W$,

$$\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V]$$

Composition of $T_1 : U \to V$ and $T_2 : V \to W$:

$$(T_2 T_1)(u) = T_2(T_1(u))$$

LT is **one-to-one** if $v_1 \neq v_2 \Rightarrow T(v_1) \neq T(v_2)$

$$T \text{ one-to-one } \Leftrightarrow \text{Ker}(T) = \{0\}$$

LT is **onto** if every $w \in W$ is the image of at least one $v \in V$

$$T \text{ onto } \Leftrightarrow \text{Rng}(T) = W$$

If $T : V \to W$ is both one-to-one and onto: **inverse LT**

$$T^{-1}(w) = v \Leftrightarrow w = T(v)$$

Isomorphism: $V \cong W$ if exists T that is 1-1 and onto
Matrix representation relative to bases B and C, for $B = \{v_1, \dots, v_n\}$ and $C = \{w_1, \dots, w_m\}$:

$$[T]_B^C = \begin{bmatrix} [T(v_1)]_C & \dots & [T(v_n)]_C \end{bmatrix}$$

Identity of matrix representation:

$$[T(v)]_C = [T]_B^C [v]_B \quad [T_2 T_1]_A^C = [T_2]_B^C [T_1]_A^B$$

Chapter 7 - Eigenvalues and eigenvectors

Eigenvalue: $Av = \lambda v$ has nontrivial solutions v
Eigenvalue/eigenvector problem:

$$\det(A - \lambda I) = 0 \Rightarrow (A - \lambda_i I) v_i = 0$$

Characteristic polynomial: $p(\lambda) = \det(A - \lambda I)$

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_k)^{m_k}$$

Algebraic multiplicity: $m_1, \dots, m_k$
Eigenspace: space spanned by v_i for each λ_i
Defective matrix: $n \times n$ matrix with $< n$ l.i. eigenvectors

$$A \text{ nondefective} \Leftrightarrow \dim[E_i] = m_i\ \forall 1 \leqslant i \leqslant k$$

A **similar to** B if exists S such that $B = S^{-1} A S$
Similar matrices have the same eigenvalues
Diagonalizable: $n \times n$ matrix similar to a diagonal matrix

Chapter 1 - First-order differential equations

Linear differential equation of **order** n:

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = F(x)$$

Existence and uniqueness theorem: let $f(x, y)$ continuous on $R : [a, b] \times [c, d]$. If $\partial f / \partial y$ continuous in R, then there exists an interval I
Slope field: sketch of $dy/dx = f(x, y)$ at different points
Separable differential equation:

$$p(y) \frac{dy}{dx} = q(x) \Rightarrow \int p(y)\ \mathrm{d}y = \int q(x)\ \mathrm{d}x + C$$

Integrating factor:

$$\frac{dy}{dx} + p(x) y = q(x) \Rightarrow I(x) = \exp\left(\int p(x)\ \mathrm{d}x\right)$$

Chapter 8 - Linear differential equations of order n

Derivative operator $D(f) = f'$:

$$Ly = D^n y + a_1 D^{n-1} y + \dots + a_{n-1} D y + a_n y$$

General solution of homogeneous differential equations:

$$y(x) = c_1 y_1(x) + \dots + c_n y_n(x)$$

Solving differential equations $Ly = F$:
Auxiliary equation:

$$P(r) = r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = (r - r_1)^{m_1} \dots (r - r_k)^{m_k} = 0$$

r_i real: $e^{r_i x}, x e^{r_i x}, \dots$
r_j complex: $e^{ax} \cos bx, x e^{ax} \cos bx, \dots, e^{ax} \sin bx, x e^{ax} \sin bx, ..$
Annihilator: $A(D) F = 0 \Rightarrow P(D) y = 0$

$$A(D) = (D - a)^{k+1} \Leftarrow (a_0 + a_1 x \dots + a_k x^k) e^{ax}$$

$$A(D) = (D^2 - 2aD + a^2 + b^2)^{k+1} \Leftarrow$$
$$(a_0 + \dots + a_k x^k) e^{ax} \cos bx + (b_0 + \dots + b_k x^k) e^{ax} \sin bx$$