MATH 225 Homework 7

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Problem 1: Goode 8.1.3

Find Ly for $L = D^3 - 2xD^2$ if $y_1(x) = 2e^{3x}$, $y_2(x) = 3\ln x$, $y_3(x) = 2e^{3x} + 3\ln x$.

Solution. For $y_1(x) = 2e^{3x}$, $D^3 - 2xD^2 = 54e^{3x} - 36xe^{3x}$. For $y_2(x) = 3\ln x$, $D^3 - 2xD^2 = 6x^{-3} + 6x^{-1}$. For $y_3 = 2e^{3x} + 3\ln x$, $D^3 - 2xD^2 = 54e^{3x} - 36xe^{3x} + 6x^{-3} + 6x^{-1}$, as L is a linear transform.

Problem 2: Goode 8.1.7

Verify that $y(x) = \sin x^2$ is in the kernel of $L = D^2 - x^{-1}D + 4x^2$.

Solution. Consider $y(x) = \sin x^2$, $Ly = D^2y - x^{-1}Dy + 4x^2y$. Substitution gives

$$2\cos x^2 - 4x^2\sin x^2 - 2\cos x^2 + 4x^2\sin x^2 = 0,$$

proving that $y(x) = \sin x^2$ is in the kernel of L.

Problem 3: Goode 8.1.12

Compute Ker(L) for $L = D^2 + 2D - 15$.

Solution. Consider $Ly = 0 \Rightarrow (D+5)(D-3)y = 0$. The two solutions is then e^{-5x} and e^{3x}, hence

$$\text{Ker}(L) = c_1e^{-5x} + c_2e^{3x}.$$

Problem 4: Goode 8.1.17

Write

$$y''' + x^2y'' - \sin xy' + e^xy = x^3$$

as an operator equation, and give the associated homogeneous differential equation.

Solution. The operator equation is $(D^3 + x^2D^2 - \sin xD + e^x)y = x^3$, and the associated homogeneous differential equation is $(D^3 + x^2D^2 - \sin xD + e^x)y = 0$.

Problem 5: Goode 8.1.21

Determine which of the following set of vectors is a basis for the solution space to the differential equation $y'' - 16y = 0$:

$$S_1 = \left\{e^{4x}\right\}, S_2 = \left\{e^{2x}, e^{4x}, e^{-4x}\right\}, S_3 = \left\{e^{4x}, e^{2x}\right\}, S_4 = \left\{e^{4x}, e^{-4x}\right\}, S_5 = \left\{e^{4x}, 7e^{4x}\right\}, S_6 = \{\cosh 4x, \sinh 4x\}.$$

Solution. Consider $(D+4)(D-4)y = 0$. It can easily be verified that e^{4x} and e^{-4x} are the set of solutions, hence S_4 are S_6 are the basis. S_1 is missing a solution, S_2 and S_3 have e^{2x} that is not a solution, and S_5 has two linearly dependent solutions. Consider the hyperbolic functions as sums of exponential functions, S_6 is a basis.

Problem 6: Goode 8.1.23

Determine two linearly independent solutions to the given differential equation of the form $y(x) = e^{rx}$, and thereby determine the general solution to the differential equation

$$y'' - 2y' - 3y' = 0.$$

Solution. The differential equation is equivalent to

$$(D+1)(D-3)y = 0,$$

which gives solutions e^{-x} and e^{3x}. The general solution is then

$$y(x) = c_1 e^{-x} + c_2 e^{3x}.$$

Problem 7: Goode 8.1.28

Determine three linearly independent solutions to the given differential equation of the form $y(x) = e^{rx}$, and thereby determine the general solution to the differential equation.

Solution. The differential equation is equivalent to

$$(D+3)(D+2)(D-2)y = 0,$$

which gives solutions e^{-2x}, e^{2x}, and e^{-3x}. The general solution is then

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}.$$

Problem 8: Goode 8.1.38

Determine a particular solution to the given differential equation of the form $y_p(x) = A_0 e^{5x}$. Also find the general solution to the differential equation

$$y'' + y' - 6y = 18e^{5x}.$$

Solution. The corresponding homogeneous equation is equivalent to

$$(D+3)(D-2)y = 0,$$

which gives solutions e^{-3x} and e^{2x}. Now consider $y_p(x)$, substitution into the nonhomogeneous differential equation gives

$$25A_0e^{5x} + 5A_0e^{5x} - 6A_0e^{5x} = 18e^{5x} \Rightarrow A_0 = 0.75.$$

The general solution is then

$$y(x) = c_1e^{-3x} + c_2e^{2x} + 0.75e^{5x}.$$

Problem 9: Goode 8.2.9

Determine the general solution to the given differential equation: $y'' - 6y' + 9y = 0$.

Solution. The homogeneous equation is equivalent to

$$(D-3)^2 = 0,$$

which gives solutions e^{3x} and xe^{3x}. The general solution is then

$$y(x) = c_1e^{3x} + c_2xe^{3x}.$$

Problem 10: Goode 8.2.10

Determine the general solution to the given differential equation: $(D^2 + 6D + 25)y = 0$.

Solution. The homogeneous equation is equivalent to

$$(D+3+4i)(D+3-4i) = 0,$$

so it has roots $-3 \pm 4i$. The general solution is then

$$y(x) = c_1e^{-3x}\cos 4x + c_2e^{-3x}\sin 4x.$$

Problem 11: Goode 8.2.22

Determine the general solution to the given differential equation: $(D-2)(D^2-16)y=0$.

Solution. The homogeneous equation is equivalent to

$$(D-2)(D+4)(D-4)=0,$$

which gives solutions e^{2x}, e^{4x}, and e^{-4x}. The general solution is then

$$y(x)=c_1e^{2x}+c_2e^{4x}+c_3e^{-4x}.$$

Problem 12: Goode 8.2.35

Solve the given initial problem: $y''-8y'+16y=0$, $y(0)=2$, $y'(0)=7$.

Solution. The homogeneous equation is equivalent to

$$(D-4)^2=0,$$

which gives solutions e^{4x} and xe^{4x}. The general solution is then

$$y(x)=c_1e^{4x}+c_2xe^{4x}.$$

Consider the initial-value problem, $c_1=2$, and $c_2=-1$.

Problem 13: Goode 8.3.9

Determine the annihilator of the given function: $F(x)=e^{-3x}(2\sin x+7\cos x)$.

Solution. Applying $A(D)=D^2-2ad+a^2+b^2$ gives $A(D)=D^2+6D+10$.

Problem 14: Goode 8.3.10

Determine the annihilator of the given function: $F(x)=e^{4x}(x-2\sin 5x)+3x-x^2e^{-2x}\cos x$.

Solution. Consider $F(x)=F_1(x)+F_2(x)+F_3(x)+F_4(x)$, so $A(D)=A_1(D)A_2(D)A_3(D)A_4(D)$. We know that $A_1(D)=(D-a)^{k+1}=(D-4)^2$, $A_2(D)=(D^2-2aD+a^2+b^2)^{k+1}=D^2-8D+41$, $A_3(D)=(D-a)^{k+1}=D^2$, $A_4=(D^2-2aD+a^2+b^2)=(D^2+4D+5)^3$. $A(D)$ is naturally the product, so

$$A(D)=(D-4)^2(D^2-8D+41)(D^2)(D^2+4D+5)^3.$$

Problem 15: Goode 8.3.22

Determine the general solution to the given differential equation: $(D^2-1)y = 3e^{2x} - 8e^{3x}$.

Solution. Solving the homogeneous case gives us $y_c(x) = c_1e^x + c_2e^{-x}$. Now we consider the annihilator $A(D) = A_1(D)A_2(D)$, where $A_1(D) = D-2$, $A_2(D) = D-3$. This gives the trial solution $y = A_0e^{2x} + A_1e^{3x}$. Now,

$$(D^2-1)y = D^2y - y = (4A_0e^{2x} + 9A_1e^{3x}) - (A_0e^{2x} + A_1e^{3x}) = 3A_0e^{2x} + 8A_1e^{3x}.$$

Equating this to $F(x)$ gives $A_0 = 1$, $A_1 = -1$. It follows that

$$y(x) = c_1e^x + c_2e^{-x} + e^{2x} - e^{3x}.$$

Problem 16: Goode 8.3.35

Solve the given initial problem: $y'' + y' - 2y = -10\sin x$, $y(0) = 2$, $y'(0) = 1$.

Solution. Solving the homoegeneous case gives us $y_c(x) = c_1e^x + c_2e^{-2x}$. Now we consider the annihilator $A(D) = D^2 + 1$. This gives the trial solution $A_0\cos x + A_1\sin x$. Now,

$$(D^2+D-2)y = -A_0\cos x - A_1\sin x - A_0\sin x + A_1\cos x - 2A_0\cos x - 2A_1\sin x = (-A_0-3A_1)\sin x + (-3A_0+A_1)\cos x.$$

Equating this to $F(x)$ gives $A_0 = 1$, $A_1 = 3$. It follows that

$$y(x) = e^{-2x} + \cos x + 3\sin x.$$