ECON 577 Homework 3

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Problem: Campbell 3.1. Assume that the Sharpe-Lintner CAPM holds, so the mean-variance efficient fron- tier consists of combinations of Treasury bills and the market portfolio. Nonetheless, some households make the mistake of holding undiversified portfolios that contain only one stock or a few stocks. (Empirical evidence on such behavior is discussed in Chapter 10.)

(a) Show that the Sharpe ratio of any portfolio divided by the Sharpe ratio of the market portfolio equals the correlation of that portfolio with the market portfolio.

(b) Suppose the market is made up of identical stocks, each of which has the same mar- ket capitalization, the same mean and variance of return, and the same correlation $\rho > 0$, with every other individual stock. Consider the limit as the number of stocks in the market increases. What is the Sharpe ratio of an equally-weighted portfolio that contains N stocks divided by the Sharpe ratio of the market portfolio? Interpret.

Sol of part (a). The Sharpe ratio is essentially the ratio of excess return over standard deviation. Therefore

$$\frac{SR_p}{SR_m} = \frac{(R_p - R_f)/\sigma_p}{(R_m - R_f)/\sigma_m}.$$

By the CAPM model, we have that, for any portfolio p, $R_p - R_f = \beta_{pm}(R_m - R_f)$, where

$$\beta_{pm} = \frac{\mathrm{Cov}(R_p, R_m)}{\mathrm{Var}(R_p, R_m)}.$$

We therefore see that

$$\frac{SR_p}{SR_m} = \frac{\mathrm{Cov}(R_p, R_m)}{\mathrm{Var}(R_p, R_m)}\frac{\sigma_m}{\sigma_p} = \frac{\rho(R_p, R_m)\sigma_m\sigma_p}{\sigma_m\sigma_m} \cdot \frac{\sigma_m}{\sigma_p} = \rho(R_p, R_m).$$

Sol of part (b). Assume there are $M \geqslant N$ stocks in the market. First we consider the case where $M \gg N$. The variance of the portfolio is

$$\mathrm{Var}(R_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}(R_i, R_j) = \frac{1}{N}\sigma^2 + \frac{N-1}{N}\mathrm{Cov}(\cdot,\cdot),$$

where all covariances are equal. Therefore we see as N increases, the share of variance σ^2 will decrease to zero, and thus $\mathrm{Var}(R_p) \to \mathrm{Cov}(\cdot,\cdot)$. Similarly, as the market could be understood as an equally weighted portfolio, it would have variance

$$\mathrm{Var}(R_m) = \frac{1}{M}\sigma^2 + \frac{M-1}{N}\mathrm{Cov}(\cdot,\cdot).$$

Because the return is the same across the board, the ratio of Sharpe ratios is simply the reciprocal of standard deviations σ_m/σ_p. That is,

$$\frac{SR_p}{SR_m} = \sqrt{\frac{N^{-1}\sigma^2 + (N-1)N^{-1}\mathrm{Cov}(\cdot,\cdot)}{M^{-1}(\sigma^2) + (M-1)M^{-1}\mathrm{Cov}(\cdot,\cdot)}}.$$

We see that as $N \to M^-$, the characteristic of the equal ratio portfolio approaches the characteristic of the market portfolio, with a portfolio variance approaching the covariance of individual pair of stocks. The equality is reached where $N = M$, where the equally-weighted portfolio equates the market portfolio; thus the ratio of Sharpe ratios is naturally unity.

Problem: Campbell 3.3. Consider a static frictionless economy with a riskless aset in zero net supply with return R_f, and n risky assets with jointly normal returns R_i, $i = 1, \cdots, n$. Suppose the risky returns obey a linear K-factor model:

$$R_i = \overline{R}_i + \sum_{k=1}^{K} b_{ik} f_k + \varepsilon_i,$$

where the factors and residuals are normally distributed, $\overline{R}_i$ and b_{ik} are constants, $\mathrm{Var}(\varepsilon_i) = \sigma_i^2$, $\mathrm{Var}(f_i) = 1$, and $\mathbb{E}[\varepsilon_i] = \mathbb{E}[f_k] = \mathrm{Cov}(\varepsilon_i, \varepsilon_k) = \mathrm{Cov}(\varepsilon_i, f_i) = \mathrm{Cov}(\varepsilon_i, f_k) = \mathrm{Cov}(f_i, f_k) = 0$ for all i and k with $k \neq i$. There is a representative investor who has exponential utility with coefficient of absolute risk aversion A. Assume that the representative investor has wealth $W = 1$.

For each asset i, define the deviation α_i from the average excess return predicted by arbitrage pricing theory as

$$\alpha_i \equiv \overline{R}_i - R_f - \sum_{k=1}^{K} b_{ij}\lambda_k,$$

where λ_k is the price of risk of factor k. Show that $\alpha_i = A w_i \sigma_i^2$, where w_i is the share of asset i in the market portfolio. Interpret this result.

Sol. Given a representative investor with exponential utility, $U(W) = -e^{-AW}$, where A is the coefficient of absolute risk aversion. Considering the end-of-period wealth, the expectation is

$$W' = (W - \theta)(1 + R_f) + \theta(1 + R_m),$$

where R_f is the risk-free rate and R_m is the return on the market portfolio. Here the investor invests θ in the risky portfolio and $W - \theta$ in bonds at the risk-free rate. Given the exponential utility the function, the investor problem wishes to maximize

$$\max_{\theta} W(1 + R_f) + \theta\left(\overline{R}_m - R_f\right) - \frac{A}{2}\theta^2\sigma_m^2.$$

Here $R_m - R_f$ is the market's excess return and σ_m^2 the market's variance.

To maximize the expression, we have that

$$\frac{\mathrm{d}\mathbb{E}[U]}{\mathrm{d}\theta} = \mathbb{E}\left[\overline{R}_m - R_f\right] - A\theta\sigma_m^2 = 0 \Rightarrow \theta^* = \frac{\mathbb{E}\left[\overline{R}_m - R_f\right]}{A\sigma_m^2}.$$

However in static equilibrium, every investor holds the market portfolio, and therefore $\theta^* = w = 1$, bringing us to $\mathbb{E}\left[\overline{R}_m - R_f\right] = A\sigma_m^2$, giving an expression for the market's excess return in terms of risk aversion and market variance. Similar to the market, we can write the relationship for the asset as $\mathbb{E}[R_i - R_f] = A\sigma_i^2 w_i$.

For a particular asset i, substituting the linear K-factor model gives us the expression of

$$\alpha_i = \overline{R}_i - R_f - \sum_{k=1}^{K} b_{ik} f_k \Rightarrow \alpha_i = A w_i \sigma_i^2.$$

The interpretation of the result is more straightforward: the deviation of the average excess return is affected by (in fact, proportional to) σ_i^2, the risk associated with asset i both systematic and idiosyncratic, as well as its weight w_i in the market portfolio, and the investor's Arrow-Pratt coefficient A.

Problem. Consider two assets i and j that follow the basic market model:

$$R_k^e = \alpha_k + \beta_k R_M^e + e_k, \quad k = i, j$$

where R_k^e is the excess return on the asset, α_k is the asset's "alpha" which is assumed to be constant, β_k is the asset's market beta, R_M^e is the excess return on the market, and e_k is a zero-mean residual term. Assume that the correlation between e_i and e_j is given by $\rho_{i,j}$. Assume $\alpha_i > 0$ and $\alpha_j < 0$. Consider a portfolio that goes long 100% asset i and short 100% asset j and assume that both assets have the same market betas and residual variances. Define the risk-to-variance ratio of the portfolio as

$$\gamma = \frac{R_P^e}{\mathrm{Var}(R_P^e)}.$$

Compute $\frac{\mathrm{d}\gamma}{\mathrm{d}\rho_{i,j}}$. What is the sign of the derivative? Explain the intuition for the sign of the derivative.

Sol. We consider the following returns:

$$\begin{cases} R_i = \alpha_i + \beta_i R_M^e + e_i + R_f, \\ R_j = \alpha_j + \beta_j R_M^e + e_j + R_f. \end{cases}$$

If we long R_i and short R_j, our portfolio will have

$$R_p = (\alpha_i - \alpha_j) + (\beta_i - \beta_j) R_M^e + e_i - e_j - R_f,$$

where the blue parts cancel to zero from the prompt. To calculate the variance of excess returns, we observe

$$\mathrm{Var}(R_p^e) = \mathrm{Var}(R_i^e) + \mathrm{Var}(R_j^e) - 2\mathrm{Cov}(R_i^e, R_j^e).$$

Considering the respective expressions, we see

$$\mathrm{Var}(R_i^e) = \beta^2 \mathrm{Var}(R_m^e) + \mathrm{Var}(e_i), \quad \mathrm{Var}(R_j^e) = \beta^2 \mathrm{Var}(R_M^e) + \mathrm{Var}(e_j),$$

and regarding the covariance,

$$\mathrm{Cov}(R_i^e, R_j^e) = \beta^2 \mathrm{Cov}(R_M^e, R_M^e) + \mathrm{Cov}(e_i, e_j).$$

The last term here equals $\rho_{ij}\sigma^2$ where σ is the standard deviation of the e_k 's. With the expressions in mind, we could then substitute in the variance expression to obtain

$$\mathrm{Var}(R_p^e) = 2\beta^2 \mathrm{Var}(R_M^e) + 2\sigma^2 - 2\beta^2 \mathrm{Var}(R_M^e) - 2\rho_{ij}\sigma^2.$$

Thus derivation gives

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\rho_{ij}} = \frac{\alpha_i - \alpha_j - R_f}{2(1-\rho_{ij})^2}.$$

Here as we assume the numerator to be positive, we see that γ takes a positive derivative against ρ_{ij}. It has a nice implication: as ρ_{ij} increases, the assets are better correlated, and therefore for the same variance more return can be obtained. In other words, for the same return desired, there is less variance, implying a better hedging method.