ECON 577 Homework 2

Stanley Hong

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Problem. Consider the exponential utility function $U = -\exp(-Aw)$. Assume the risk-free rate is zero and normalize initial wealth to $w_0 = 1$. There are two normally distributed risky assets with expected returns and volatilities (μ_1, σ_1) and (μ_2, σ_2), respectively and correlation equal to ρ. If weights must sum to one, compute the allocations w_1 and w_2 to the risky assets expressed in terms of the respective model parameters. Would your answer change if $w_0 = 1000000$? Explain why or why not.

Sol. We wish to maximize the following quantity:

$$\max_w \mathbb{E}\left[U(w)\right] = \mathbb{E}\left[-\exp(-Aw)\right].$$

Here w is the portfolio with return $R_w = w_1\mu_1 + w_2\mu_2$, where w_1 and w_2 are the respective weights. Although w remains a normal distribution, their variance takes

$$\mathrm{Var}\ w = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho.$$

Considering the expectation of the $\log$-normal, we wish to minimize

$$\mathbb{E}\left[e^{-Aw}\right] = \exp\left(-A(w_1\mu_1 + w_2\mu_2) + \frac{A^2}{2}(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho)\right).$$

Now we could consider $(\cdot) = \ln(\exp(\cdot))$ instead, and consider, subject to constraint, $w_2 = 1 - w_1$. In this case the first-order condition gives

$$-A\mu_1 + A\mu_2 + A^2\left[w_1\sigma_1^2 - (1-w_1)\sigma_2^2 + \sigma_{12} - 2w_1\sigma_{12}\right] = 0.$$

Solving for w_1 gives

$$w_1 = \frac{\mu_1 - \mu_2 + A\sigma_2^2 - A\sigma_1\sigma_2\rho}{A\sigma_1^2 + A\sigma_2^2 - 2A\sigma_1\sigma_2\rho}.$$

Subsequently $w_2 = 1 - w_1$.

Certainly, the allocations w_1 and w_2 *will* differ if $w_0 = 1000000$. However, note that it would not be a proportional change. This is because we are specifically dealing with a CARA utility function. Specifically if $w_0 = 1000000$ we will have

$$w_1 = \frac{\mu_1 - \mu_2 + 1000000\sigma_2^2 - 1000000A\sigma_1\sigma_2\rho}{A\sigma_1^2 + A\sigma_2^2 - 2A\sigma_1\sigma_2\rho},$$

an disproportional increase in investment with respect to wealth that is dependent mostly on covariance and risk averse coefficient instead of expected value difference as in the previous case.

Problem. Consider the N-asset general case for portfolio variance $\sigma_p^2 = w'\Sigma w$ where w is the vector of weights and Σ is the positive semidefinite covariance matrix of asset returns. The **marginal contribution to risk** of an asset i is defined as $MCR(i) = \frac{\partial \sigma_p}{\partial w_i}$. Prove that the portfolio standard variation is equal to the sum-product of portfolio weights and marginal contributions to risks:

$$\sigma_p = \sum_{i=1}^{N} w_i \frac{\partial \sigma_p}{\partial w_i}.$$

Proof. The statement is equivalent to

$$\sigma_p \stackrel{?}{=} w^T \begin{bmatrix} \partial\sigma_p/\partial w_1 \\ \partial\sigma_p/\partial w_2 \\ \vdots \\ \partial\sigma_p/\partial w_n \end{bmatrix} = w^T \begin{bmatrix} MCR_1 \\ MCR_2 \\ \vdots \\ MCR_n \end{bmatrix}.$$

By the chain rule, we see that

$$MCR_i = \frac{\partial \sigma_p}{\partial w_i} = \frac{1}{2\sigma_p}\frac{\partial \sigma_p^2}{\partial w_i}.$$

Here the partial derivative $\partial\sigma_p^2/\partial w_i$ is the i-th column element of the derivative $(w^T\Sigma w)$ against w. As such, we can write that, for each $MCR_i = \partial\sigma_p/\partial w_i$,

$$MCR_i = \frac{1}{2\sigma_p}(2\Sigma w)_i = \frac{(\Sigma w)_i}{\sigma_p}.$$

Now considering the product $\sum_{i=1}^{n} w_i MCR_i$, we could multiply w^T on both sides of the previous equation:

$$w^T\mathbf{MCR} = w^T\frac{(\Sigma w)}{\sigma_p} = \frac{\sigma_p^2}{\sigma_p} = \sigma_p,$$

as desired. □

Problem. The **risk parity** portfolio is a portfolio in which the marginal contribution to risk for all assets are the same. Specifically, portfolio weights are set such that $MCR(i) = MCR(j)$ for every i, j. Portfolio managers generally hold the risk parity portfolio when they do not have explicit views on assets or factors. Consider a simple two-asset universe with volatilities of σ_1 and σ_2 and correlation ρ. Derive the risk parity weights on the two assets.

Sol. For risk parity we require $MCR_1 = MCR_2$. In a two-asset case, we can directly calculate.

$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}};$$

taking derivative gives

$$\begin{cases} \partial\sigma_p/\partial w_1 = (w_1\sigma_1^2 + w_2\sigma_{12})/\sigma_p; \\ \partial\sigma_p/\partial w_2 = (w_2\sigma_2^2 + w_1\sigma_{12})/\sigma_p. \end{cases}$$

Equating the two partial derivatives give $w_1\sigma_1^2 + w_2\sigma_{12} = w_2\sigma_2^2 + w_1\sigma_{12}$. Now imposing the constraint $w_1 + w_2 = 1$, we can obtain

$$w_1\sigma_1^2 + (1 - w_1)\sigma_{12} = (1 - w_1)\sigma^2 + w_1\sigma_{12} \Rightarrow w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}},$$

where $\sigma_{12} = \sigma_1\sigma_2\rho_{12}$.

Problem. Show that the variance-minimizing portfolio weights are N^{-1} when weights are constrained to sum to one and volatilities across assets and correlations are the same across all asset pairs. Formally, $\sigma_i = \sigma_j = \sigma$ and $\rho_{i,j} = \rho$ for every i, j. Derive an expression for the variance of the minimum variance portfolio as a function of σ and ρ. Is it reasonable to from an economics perspective to assume that $\rho_{i,j} = 0$ for every i, j? Why or why not?

Sol. We first give a closed-form expression to portfolio variance. Specifically, we have that

$$\sigma_p^2 = \sigma^2\left(\sum_{i=1}^{N} w_i^2 + \sum_{i=1}^{N}\sum_{i\neq j} w_i w_j \rho\right) = \sigma^2\left(\sum_{i=1}^{N} w_i^2 + \rho(1 - \sum_{i=1}^{N} w_i^2)\right),$$

with the second equality possible as w_i 's sum to one. Now our only goal here is to minimize the sum of squared term $\sum w_i^2$ subject to $\sum w_i = 1$, which is trivially $w_i = N^{-1}$. More specifically, the Lagrangian takes

$$2w_i - \lambda = 0 \Rightarrow w_i = w_j \ \forall i \neq j.$$

The overall variance is $w^T\Sigma w$ which equals

$$\sigma_p^2 = \frac{(N - 1 + \rho)\sigma^2}{N}.$$

It wouldn't be economic to think $\rho = 0$ across all assets in the portfolio. This is due to the fact that assets in real life are often closely correlated, sometimes as complements or substitutes. There could possibly be a pair of assets that are not as related, but one should refrain themselves from generalizing to *every* pair of assets.