MATH 225 Homework 5

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Problem 1: Goode 6.1.2

Verify that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + 2x_2, 2x_1 - x_2)$ is a linear transformation.

Solution. Consider $a = (a_1, a_2)$ and $b = (b_1, b_2)$. $T(a + b) = ((a_1 + b_1) + 2(a_2 + b_2), 2(a_1 + b_1) - (a_2 + b_2)) = (a_1 + 2a_2 + b_1 + 2b_2, 2a_1 - a_2 + 2b_1 - b_2) = (a_1 + 2a_2, 2a_1 - a_2) + (b_1 + 2b_2, 2b_1 - b_2) = T(a) + T(b)$.
Now consider $a = (a_1, a_2)$ and $c \in \mathbb{R}$. $T(ca) = (ca_1 + 2ca_2, 2ca_1 - ca_2) = c(a_1 + 2a_2, 2a_1 - a_2) = cT(a)$.
The linearity properties are satisfied, so T is a linear transformation.

Problem 2: Goode 6.1.4

Verify that $T : C^2(I) \to C^0(I)$ defined by $T(y) = y'' - 16y$ is a linear transformation.

Solution. Consider $f, g \in C^2(I)$. $T(f + g) = (f + g)'' - 16(f + g) = f'' - 16f + g'' - 16g = T(f) + T(g)$.
Now consider $f \in C^2(I)$. $T(cf) = (cf)'' - 16(cf) = cf'' - c(16f) = c(f'' - 16f) = cT(f)$.
The linearity properties are satisfied, so T is a linear transformation.

Problem 3: Goode 6.1.6

Verify that $T : C^0[a, b] \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\,\mathrm{d}x$ is a linear transformation.

Solution. Consider $f, g \in C^0[a, b]$. $T(f + g) = \int_a^b f(x) + g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x = T(f) + T(g)$.
Now consider $f \in C^0[a, b]$. $T(cf) = \int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x = cT(f)$.
The linearity properties are satisfied, so T is a linear transformation.

Problem 4: Goode 6.1.16

Determine the matrix of the given transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_3 - x_1)$.

Solution. $T(e_1) = (1, -1)$, $T(e_2) = (-1, 0)$, $T(e_3) = (1, 1)$. The matrix is then $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Problem 5: Goode 6.1.24

Let V be a real inner product space, and let u be a fixed (nonzero) vector in V. Define $T : V \to \mathbb{R}$ by $T(v) = \langle u, v\rangle$. Use properties of the inner product to show that T is a linear transformation.

Solution. $T(v+w) = \langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle = T(v) + T(w)$, and $T(kv) = \langle u, kv\rangle = k\langle u, v\rangle = kT(v)$. The linearity properties are satisfied, so T is a linear transformation.

Problem 6: Goode 6.1.30

Assume that T defines a linear transformation, $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that $T(0,-1,4) = (2,5,-2,1)$, $T(0,3,3) = (-1,0,0,5)$, and $T(4,4,-1) = (-3,1,1,3)$. Find the matrix of T.

Solution. Consider $A^{\#} = \begin{bmatrix} 0 & -1 & 4 & 2 & 5 & -2 & 1 \\ 0 & 3 & 3 & -1 & 0 & 0 & 5 \\ 4 & 4 & -1 & -3 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3/2 & -1/4 & -1/4 \\ 0 & 1 & 0 & -2/3 & -1 & 2/5 & 17/15 \\ 0 & 0 & 1 & 1/3 & 1 & -2/5 & 8/15 \end{bmatrix}$. This gives $e_1 = (0, 3/2, -1/4, -1/4)$, $e_2 = (-2/3, -1, 2/5, 17/15)$, $e_3 = (1/3, 1, -2/5, 8/15)$. Hence, the matrix of T is

$$A = [T(e_1), T(e_2), T(e_3)] = \begin{bmatrix} 0 & -2/3 & 1/3 \\ 3/2 & -1 & 1 \\ -1/4 & 2/5 & -2/5 \\ -1/4 & 17/15 & 8/15 \end{bmatrix}.$$

Problem 7: Goode 6.2.2

For the transformation of $\mathbb{R}^2$ with the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, sketch the transform of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, $(1,2)$.

Solution. Any point $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ becomes $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$ under the transformation. This is essentially rotating any point by $\pi/2$ clockwise around the origin. Hence, the square is unchanged in side length or area, and the vertices now become $(1,-1)$, $(1,-2)$, $(2,-2)$, $(2,-1)$.

Problem 8: Goode 6.2.3

For the transformation of $\mathbb{R}^2$ with the matrix $A = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix}$, sketch the transform of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, $(1,2)$.

Solution. Any point $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ becomes $\begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 - 2x_2 \\ -2x_1 \end{bmatrix}$ under the transformation. The square becomes a triangle, and the vertices now become $(-4,-2)$, $(-6,-4)$, $(-8,-4)$, $(-6,-2)$.

Problem 9: Goode 6.2.9

Describe the transformation of $\mathbb{R}^2$ with the given matrix as a product of reflections, stretches, and shears.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution.

$$A \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The operations are $A_{12}(-3)$, $A_{21}(1)$, $M_2(-1/2)$. Hence, $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

We see that T consists of three operations: a stretch in the $-y$ direction, a shear parallel to the x-axis, and a shear parallel to the y-axis.

Problem 10: Goode 6.2.10

Describe the transformation of $\mathbb{R}^2$ with the given matrix as a product of reflections, stretches, and shears.

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 8 \end{bmatrix}$$

Solution.

$$A \sim \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The operations are $A_{12}(-2)$, $A_{21}(3/2)$, $M_2(1/2)$. Hence, $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

We see that T consists of three operations: a stretch in the y-direction, a shear parallel to the x-axis, and a shear parallel to the y-axis.

Problem 11: Goode 6.3.2

Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x) = Ax$, where $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix}$. For each x below, find $T(x)$ and thereby determine whether x is in Ker(T).

(a) $x = (7, 5, -1)$

(b) $x = (-21, -15, 2)$

(c) $x = (35, 25, -5)$

Solution. (a) $T(x) = Ax = (0, 0)$, $x \in \text{Ker}(T)$.

(b) $T(x) = Ax = (-2, -3)$, $x \notin \text{Ker}(T)$.

(c) $T(x) = Ax = (0, 0)$, $x \in \text{Ker}(T)$.

Problem 12: Goode 6.3.4

Given $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x) = Ax$, where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, find Ker($T$) and Rng($T$), and give a geometrical description of each. Also, find $\dim[\text{Ker}(T)]$ and $\dim[\text{Rng}(T)]$, and verify theorem 6.3.8.

Solution. For Ker(T), we want the solution set of $Ax = 0$. As $\det(A) = 3 - 4 = -1$, the homogeneous system has only the trivial solution, so $\text{Ker}(T) = \{0\}$, and its dimension is zero. $\text{Rng}(T) = \text{colspace}(A) = \mathbb{R}^3$, and its dimension is 3. The general r-n theorem can then be verified as $0 + 3 = 3 = \dim[V]$.

Problem 13: Goode 6.3.21

Let $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$ be bases for real vector spaces V and W, respectively, and let $T : V \to W$ be the linear transformation satisfying

$$T(v_1) = 2w_1 - w_2 \quad T(v_2) = w_1 - w_2 \quad T(v_3) = w_1 + 2w_2.$$

Find Ker(T), Rng(T), and their dimensions.

Solution. For $v \in V$, there exists scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that $v = c_1v_1 + c_2v_2 + c_3v_3$. Now we have

$$T(v) = T(c_1v_1 + c_2v_2 + c_3v_3) = c_1(2w_1 - w_2) + c_2(w_1 - w_2) + c_3(w_1 + 2w_2) = (2c_1 + c_2 + c_3)w_1 + (-c_1 - c_2 + 2c_3)w_2.$$

Consider $T(v) = 0$ gives the system of equation $2c_1 + c_2 + c_3 = -c_1 - c_2 + 2c_3 = 0 \Rightarrow c_1 + 3c_3 = 0$. Setting c_3 as a free variable gives $c_1 = -3c_3$ and $c_2 = 5c_3$. Hence, $\text{Ker}(T) = \{r(-3, 5, 1), r \in \mathbb{R}\}$, and its dimension is 1.
Now consider the range. As $2c_1 + c_2 + c_3$ and $-c_1 - c_2 + 2c_3$ are linearly independent, the range is the entirety of W, and its dimension is 2.

Problem 14: Goode 6.4.3

Consider $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformations with matrices

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find T_1T_2, Ker(T_1T_2), Rng(T_1T_2), T_2T_1, Ker(T_2T_1), and Rng(T_2T_1).

Solution. $T_1T_2 = \begin{bmatrix} -1 & -8 & 5 \\ 1 & 0 & 1 \\ 1 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -8 & 6 \\ 0 & 0 & 0 \end{bmatrix}$. $\text{Ker}(T_1T_2) = \{r(-1, 4/3, 1), r \in \mathbb{R}\}$, $\text{Rng}(T_1T_2) = s(-1, 1, 1) + t(2, 0, 1)$. $T_2T_1 = \begin{bmatrix} 3 & -1 \\ 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$. $\text{Ker}(T_2T_1) = \{0\}$, $\text{Rng}(T_2T_1) = \mathbb{R}^2$.

Problem 15: Goode 6.4.12

For $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$, find Ker$(T)$ and Rng(T), and hence, determine whether the given transformation is one-to-one, onto, both, or neither. If T^{-1} exists, find it.

Solution. Ker$(T) = \{r(-2,1), r \in \mathbb{R}\}$. Rng$(T)$ = span $\{(1,-2)\}$. The kernel has more than one element, so the transformation is not one-to-one. The range is not the entirety of $\mathbb{R}^2$, so the transformation is not onto. T^{-1} does not exist.

Problem 16: Goode 6.4.24

Let v_1 and v_2 be a basis for the vector space V, and suppose that $T_1 : V \to V$ and $T_2 : V \to V$ are the linear transformations satisfying

$$T_1(v_1) = v_1 + v_2 \quad T_1(v_2) = v_1 - v_2 \quad T_2(v_1) = (v_1 + v_2)/2 \quad T_2(v_2) = (v_1 - v_2)/2.$$

Find $(T_1T_2)(v)$ and $(T_2T_1)(v)$ for an arbitrary vector in V and show that $T_2 = T_1^{-1}$.

Solution. Consider $v \in V$ that can be represented as $v = c_1v_1 + c_2v_2$. Hence, $(T_1T_2)(v) = T_1(T_2v) = T_1(T_2(c_1v_1 + c_2v_2)) = T_1(c_1T_2(v_1) + c_2T_2(v_2)) = T_1(c_1(v_1 + v_2)/2 + c_2(v_1 - v_2)/2) = T_1(c_1v_1/2 + c_2v_1/2 + c_1v_2/2 - c_2v_2/2) = c_1(2v_1)/2 + (2v_2)c_2/2 = c_1v_1 + c_2v_2 = v$. Also, (T_2T_1)

Problem 17: Goode 6.4.25

Determine an isomorphism between $\mathbb{R}^2$ and the vector space $P_1(\mathbb{R})$.

Solution. Consider $T(a,b) = a + bx$, which transforms a 2-tuple to a linear function.

Problem 18: Goode 6.5.4

Consider $T : \mathbb{R}^3 \to$ span $\{\cos x, \sin x\}$ given by

$$T(a,b,c) = (a - 2c)\cos x + (3b + c)\sin x.$$

Determine the matrix representation $[T]_B^C$ for the given linear transformation T and bases B and C.

(a) $B = \{(1,0,0), (0,1,0), (0,0,1)\}$, $C = \{\cos x, \sin x\}$

(b) $B = \{(2,-1,-1), (1,3,5), (0,4,-1)\}$, $C = \{\cos x - \sin x, \cos x + \sin x\}$

Problem 19: Goode 6.5.7

Consider $T : V \to V$, where V = span $\left\{e^{2x}, e^{-3x}\right\}$ given by $T(f) = f'$. Determine the matrix representation $[T]_B^C$ for the given linear transformation T and bases B and C.

(a) $B = C = \left\{e^{2x}, e^{-3x}\right\}$

(b) $B = \left\{e^{2x} - 3e^{-3x}, 2e^{-3x}\right\}$, $C = \left\{e^{2x} + e^{-3x}, -e^{2x}\right\}$