Homework 1.

1.1 - 4, 6, 9, 10.

For Problems 1–4 determine the order of the differential equation.

1. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = e^x$.

2. $\left(\dfrac{dy}{dx}\right)^3 + y^2 = \sin x$.

3. $y'' + xy' + e^x y = y'''$.

4. $\sin(y'') + x^2 y' + xy = \ln x$.

4. second-order differential equation

6. Verify that $y(x) = x/(x+1)$ is a solution to the differential equation

$$y + \frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{x^3 + 2x^2 - 3}{(1+x)^3}.$$

$y(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1} \Rightarrow y' = \frac{1}{(x+1)^2} \Rightarrow y'' = -\frac{2}{(x+1)^3}$

$LHS = y - \frac{2}{(x+1)^3} = \frac{x(x+1)^2 - 2}{(x+1)^3} = \frac{x^3+2x^2+x-2}{(x+1)^3}$

$RHS = \frac{1}{(x+1)^2} + \frac{x^3+2x^2-3}{(x+1)^3} = \frac{x^3+2x^2+x-2}{(x+1)^3}$

$LHS = RHS$

9. A glass of water whose temperature is 50°F is taken outside at noon on a day whose temperature is constant at 70°F. If the water's temperature is 55°F at 2 p.m., do you expect the water's temperature to reach 60°F before 4 p.m. or after 4 p.m.? Use Newton's law of cooling to explain your answer.

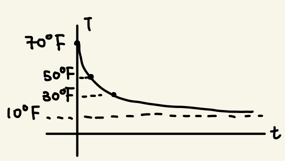
According to Newton's law of cooling, the rate of temperature change is proportional to the difference in temperature

An initial difference of 20°F can result in a 5°F change over 2 hours

$\Rightarrow$ An initial difference of 15°F will result in a $\Delta T < 5$°F change for 2 hours

$\Rightarrow$ I expect temperature to reach 60°F after 4 p.m.

10. On a cold winter day (10°F), an object is brought outside from a 70°F room. If it takes 40 minutes for the object to cool from 70°F to 30°F, did it take more or less than 20 minutes for the object to reach 50°F ? Use Newton's law of cooling to explain your answer.



Initially, $\left|\frac{dT}{dt}\right|$ is very large, according to Newton's law of cooling. Then $\left|\frac{dT}{dt}\right|$ decreases over time as the difference in temperature decreases. Hence, one can expect the T-t curve to be as shown on the left.

According to the graph, it must take less time for T to drop to 50°F because the initial rate is high, compared to dropping to 30°F.

1.2 - 3, 6, 11, 15, 19, 23, 33, 40, 44

For Problems 1–6, determine whether the differential equation is linear or nonlinear.

1. $\dfrac{d^2y}{dx^2} + e^x\dfrac{dy}{dx} = x^2$.

2. $\dfrac{d^3y}{dx^3} + 4\dfrac{d^2y}{dx^2} + \sin x\dfrac{dy}{dx} = xy^2 + \tan x$.

3. $yy'' + x(y') - y = 4x\ln x$.

4. $\sin x \cdot y'' + y' - \tan y = \cos x$.

5. $\dfrac{d^4y}{dx^4} + 3\dfrac{d^2y}{dx^2} = x$.

6. $\sqrt{x}y'' + \dfrac{1}{y'}\ln x = 3x^3$.

For Problems 7–21, verify that the given function is a solution to the given differential equation (c_1 and c_2 are arbitrary constants), and state the maximum interval over which the solution is valid.

7. $y(x) = c_1e^{-5x} + c_2e^{5x}$, $\quad y'' - 25y = 0$.

8. $y(x) = c_1\cos 2x + c_2\sin 2x$, $\quad y'' + 4y = 0$.

9. $y(x) = c_1e^x + c_2e^{-2x}$, $\quad y'' + y' - 2y = 0$.

10. $y(x) = \dfrac{1}{x+4}$, $\quad y' = -y^2$.

11. $y(x) = c_1x^{1/2}$, $\quad y' = \dfrac{y}{2x}$.

12. $y(x) = e^{-x}\sin 2x$, $\quad y'' + 2y' + 5y = 0$.

13. $y(x) = c_1\cosh 3x + c_2\sinh 3x$, $\quad y'' - 9y = 0$.

14. $y(x) = c_1x^{-3} + c_2x^{-1}$, $\quad x^2y'' + 5xy' + 3y = 0$.

15. $y(x) = c_1x^2\ln x$, $\quad x^2y'' - 3xy' + 4y = 0$.

16. $y(x) = c_1x^2\cos(3\ln x)$, $\quad x^2y'' - 3xy' + 13y = 0$.

17. $y(x) = c_1x^{1/2} + 3x^2$, $\quad 2x^2y'' - xy' + y = 9x^2$.

18. $y(x) = c_1x^2 + c_2x^3 - x^2\sin x$, $x^2y'' - 4xy' + 6y = x^4\sin x$.

19. $y(x) = c_1e^{ax} + c_2e^{bx}$, $\quad y'' - (a+b)y' + aby = 0$, where a and b are constants and $a \neq b$.

For Problems 22–25, determine all values of the constant r such that the given function solves the given differential equation.

22. $y(x) = e^{rx}$, $\quad y'' - y' - 6y = 0$.

23. $y(x) = e^{rx}$, $\quad y'' + 6y' + 9y = 0$.

24. $y(x) = x^r$, $\quad x^2y'' + xy' - y = 0$.

25. $y(x) = x^r$, $\quad x^2y'' + 5xy' + 4y = 0$.

For Problems 33–36, find the general solution to the given differential equation and the maximum interval on which the solution is valid.

33. $y' = \sin x$.

34. $y' = x^{-2/3}$.

35. $y'' = xe^x$.

36. $y'' = x^n$, n an integer.

For Problems 37–40, solve the given initial-value problem.

37. $y' = x^2\ln x$, $\quad y(1) = 2$.

38. $y'' = \cos x$, $\quad y(0) = 2$, $\quad y'(0) = 1$.

39. $y''' = 6x$, $\quad y(0) = 1$, $\quad y'(0) = -1$, $\quad y''(0) = 4$.

40. $y'' = xe^x$, $\quad y(0) = 3$, $\quad y'(0) = 4$.

41. Prove that the general solution to $y'' - y = 0$ on any interval I is $y(x) = c_1e^x + c_2e^{-x}$.

A second-order differential equation together with two auxiliary conditions imposed at different values of the independent variable is called a **boundary-value problem**. For Problems 42–43, solve the given boundary-value problem.

42. $y'' = e^{-x}$, $\quad y(0) = 1$, $\quad y(1) = 0$.

43. $y'' = -2(3 + 2\ln x)$, $\quad y(1) = y(e) = 0$.

44. The differential equation $y'' + y = 0$ has the general solution $y(x) = c_1\cos x + c_2\sin x$.

(a) Show that the boundary-value problem $y'' + y = 0$, $\; y(0) = 0$, $\; y(\pi) = 1$ has no solutions.

(b) Show that the boundary-value problem $y'' + y = 0$, $\; y(0) = 0$, $\; y(\pi) = 0$ has an infinite number of solutions.

3. DE is nonlinear because yy'' has a nonlinear coefficient

6. DE is nonlinear because $\frac{1}{y'}$ is nonlinear

11. $LHS = y' = \frac{1}{2}C_1x^{-\frac{1}{2}}$,

$RHS = \frac{y}{2x} = \frac{C_1x^{\frac{1}{2}}}{2x} = \frac{1}{2}C_1x^{-\frac{1}{2}}$

$LHS = RHS$, solution is valid for $x \in (0, +\infty)$

15. $y(x) = C_1x^2\ln x \Rightarrow y' = C_1x + C_12x\ln x$

$y'' = C_1 + 2C_1\ln x + 2C_1$

$LHS = x^2y'' - 3xy' + 4y$

$= 3C_1x^2 + 2C_1x^2\ln x - 3x^2C_1 - 6x^2C_1\ln x$

$+ 4x^2C_1\ln x = 0 = RHS$

Solution is valid for $x \in (0, +\infty)$

19. $y(x) = c_1e^{ax} + c_2e^{bx} \Rightarrow y' = ac_1e^{ax} + bc_2e^{bx}$

$\Rightarrow y'' = a^2c_1e^{ax} + b^2c_2e^{bx}$

$LHS = y'' - (a+b)y' + aby$

$= a^2c_1e^{ax} + b^2c_2e^{bx}$

$- a^2c_1e^{ax} - abc_2e^{bx} - b^2c_2e^{bx} - abc_1e^{ax}$

$+ abc_1e^{ax} + abc_2e^{bx} = 0 = RHS$

Solution is valid for $x \in \mathbb{R}$.

23. $y'' + 6y' + 9y = 0$, $y = e^{rx}$

$r^2e^{rx} + 6re^{rx} + 9e^{rx} = 0$

$\Rightarrow e^{rx}(r^2 + 6r + 9) = 0$

$\Rightarrow e^{rx} = 0$ or $r^2 + 6r + 9 = (r+3)^2 = 0$

$r = -3$. $y = e^{-3x}$

33. $\frac{dy}{dx} = \sin x \Rightarrow \int dy = \int \sin x\, dx$

$\Rightarrow y = -\cos x + C$

Solution is valid for $x \in \mathbb{R}$

40. $y'' = xe^x \Rightarrow y' = \int \underset{u}{x}\,\underset{dv}{e^x}\, dx = xe^x - e^x + C_1$

$y'(0) = 4 \Rightarrow 4 = 0 - 1 + C_1 \Rightarrow C_1 = 5$

$y' = xe^x - e^x + 5 \Rightarrow y = \int xe^x - e^x + 5\, dx$

$\Rightarrow y = xe^x - 2e^x + 5x + C_2$

$y(0) = 3 \Rightarrow 3 = 0 - 2 + 0 + C_2 \Rightarrow C_2 = 5$

$y = xe^x - 2e^x + 5x + 5$

44. $y(x) = C_1\cos x + C_2\sin x$

(a) $\left.\begin{array}{l} y(0) = 0 \rightarrow 0 = C_1 \\ y(\pi) = 1 \rightarrow 1 = C_1 \end{array}\right\}$ No solution, as $0 \neq 1$.

(b) $\left.\begin{array}{l} y(0) = 0 \rightarrow 0 = C_1 \\ y(\pi) = 0 \rightarrow 0 = C_1 \end{array}\right\}$ $C_1 = 0$, C_2 can take any number, i.e. $C_2 \in \mathbb{R}$.

Solution set becomes

$\{(C_1, C_2) \mid C_1 = 0, C_2 \in \mathbb{R}\}$, so it has infinitely many sets of solutions.

1.3 - 5, 13, 15, 31

For Problems 1–8, determine the differential equation giving the slope of the tangent line at the point (x, y) for the given family of curves.

1. $y = ce^{2x}$.

2. $y = e^{cx}$.

3. $y = cx^2$.

4. $y = c/x$.

5. $y^2 = cx$.

6. $x^2 + y^2 = 2cx$.

7. $(x - c)^2 + (y - c)^2 = 2c^2$.

8. $2cy = x^2 - c^2$.

5. $y^2 = cx$

$2yy' = c \Rightarrow y' = \frac{c}{2x} \Rightarrow y' = \frac{y}{2x}$

13. Prove that the initial-value problem

$$y' = x\sin(x + y), \quad y(0) = 1$$

has a unique solution.

$\partial f/\partial y = x\cos(x+y)$ is continuous on $x \in \mathbb{R}$

So by the existance and uniqueness theorem, there exists an interval I s.t. the DE has a unique solution

15. Do you think that the initial-value problem

$$y' = xy^{1/2}, \quad y(0) = 0$$

has a unique solution? Justify your answer.

$\frac{\partial f}{\partial y} = \frac{1}{2}xy^{-\frac{1}{2}}$, which is continuous on $y \neq 0$.

As initial value is $y(0)=0$, a rectangle that doesn't intersect $y=0$ cannot be constructed, so DE has more than one solution

For Problems 31–36, determine the slope field and some representative solution curves for the given differential equation.

31. ◇ $y' = -2xy$.

32. ◇ $y' = \dfrac{x\sin x}{1 + y^2}$.

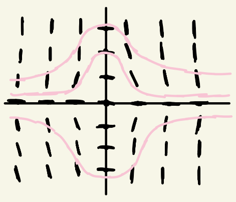
33. ◇ $y' = 3x - y$.

34. ◇ $y' = 2x^2 \sin y$.

35. ◇ $y' = \dfrac{2 + y^2}{3 + 0.5x^2}$.

36. ◇ $y' = \dfrac{1 - y^2}{2 + 0.5x^2}$.

31.



1.4 - 2, 4, 12, 16, 22

For Problems 1–11, solve the given differential equation.

1. $\dfrac{dy}{dx} = 2xy$.

2. $\dfrac{dy}{dx} = \dfrac{y^2}{x^2 + 1}$.

3. $e^{x+y}dy - dx = 0$.

4. $\dfrac{dy}{dx} = \dfrac{y}{x\ln x}$.

5. $ydx - (x - 2)dy = 0$.

6. $\dfrac{dy}{dx} = \dfrac{2x(y - 1)}{x^2 + 3}$.

7. $y - x\dfrac{dy}{dx} = 3 - 2x^2\dfrac{dy}{dx}$.

8. $\dfrac{dy}{dx} = \dfrac{\cos(x - y)}{\sin x \sin y} - 1$.

9. $\dfrac{dy}{dx} = \dfrac{x(y^2 - 1)}{2(x - 2)(x - 1)}$.

10. $\dfrac{dy}{dx} = \dfrac{x^2 y - 32}{16 - x^2} + 2$.

11. $(x - a)(x - b)y' - (y - c) = 0$, where a, b, c are constants, with $a \neq b$.

In Problems 12–15, solve the given initial-value problem.

12. $(x^2 + 1)y' + y^2 = -1, \quad y(0) = 1$.

13. $(1 - x^2)y' + xy = ax, \quad y(0) = 2a$, where a is a constant.

14. $\dfrac{dy}{dx} = 1 - \dfrac{\sin(x + y)}{\sin y \cos x}, \quad y(\pi/4) = \pi/4$.

15. $y' = y^3 \sin x, \quad y(0) = 0$.

16. One solution to the initial-value problem

$$\frac{dy}{dx} = \frac{2}{3}(y - 1)^{1/2}, \quad y(1) = 1$$

is $y(x) = 1$. Determine another solution to this initial-value problem. Does this contradict the existence and uniqueness theorem (Theorem 1.3.2)? Explain.

22. The differential equation governing the velocity of an object is

$$\frac{dv}{dt} = -kv^n,$$

where $k > 0$ and n are constants. At $t = 0$, the object is set in motion with velocity v_0. Assume $v_0 > 0$.

(a) Show that the object comes to rest in a finite time if and only if $n < 1$, and determine the maximum distance travelled by the object in this case.

(b) If $1 \leq n < 2$, show that the maximum distance travelled by the object in a finite time is less than

$$\frac{v_0^{2-n}}{(2 - n)k}.$$

(c) If $n \geq 2$, show that there is no limit to the distance that the object can travel.

2. $\frac{dy}{dx} = \frac{y^2}{x^2+1}$

$\Rightarrow \frac{1}{y^2}dy = \frac{1}{x^2+1}dx$

$\Rightarrow -\frac{1}{y} = \tan^{-1}x + C$

$\Rightarrow y = -\frac{1}{\tan^{-1}x + C}$

4. $\frac{dy}{dx} = \frac{y}{x\ln x}$

$\Rightarrow \frac{1}{y}dy = \frac{1}{x\ln x}dx \quad \ln x = u, \frac{1}{x}dx = du$

$\Rightarrow \ln y = \ln(\ln x) + C$

$y = c\ln x$

12. $\frac{dy}{dx} = \frac{-1-y^2}{x^2+1}$

$\Rightarrow \frac{1}{x^2+1}dx = -\frac{1}{1+y^2}dy$

$\Rightarrow \tan^{-1}x = -\tan^{-1}y + C$

$\Rightarrow \tan^{-1}y = -\tan^{-1}x + C$

$\Rightarrow y = \tan(-\tan^{-1}x + C)$

$y(0) = 1 \Rightarrow 1 = \tan(0 + C) \Rightarrow C = \pi n + \frac{\pi}{4}, n \in \mathbb{Z}$

$\Rightarrow y = \tan(-\tan^{-1}x + n\pi + \frac{\pi}{4}), n \in \mathbb{Z}$

16. $\frac{dy}{dx} = \frac{2}{3}(y-1)^{\frac{1}{2}} \Rightarrow \frac{dy}{(y-1)^{\frac{1}{2}}} = \frac{2}{3}x\,dx$

$\Rightarrow 2(y-1)^{\frac{1}{2}} = \frac{1}{3}x^2 + C$

$\Rightarrow y - 1 = (\frac{1}{6}x^2 + C)^2$

$\Rightarrow y = (\frac{1}{6}x^2 - \frac{1}{6})^2 + 1$

$\Rightarrow y = \frac{1}{36}x^4 - \frac{1}{18}x^2 + \frac{37}{36}$.

22. (a) $\frac{dv}{v^n} = -k\,dt$

$-\frac{1}{(n-1)v^{n-1}} = -kt + C_1 \Rightarrow C_1 = -\frac{1}{(n-1)v_0^{n-1}}$

$\frac{1}{(n-1)v^{n-1}} = kt + \frac{1}{(n-1)v_0^{n-1}}, k > 0$

$v = ((n-1)kt + v_0^{1-n})^{\frac{1}{1-n}}$

$v = 0$. If $n \geq 1, (n-1)kt \geq 0$, so $v > 0\ \forall t > 0$.

This implies n must smaller than 1

$t_0 = \frac{v_0^{1-n}}{(1-n)k}$

$S = \int_0^{t_0} v(t)\,dt = \left[\frac{1}{(n-2)k}((n-1)kt + v_0^{(1-n)})^{\frac{1}{1-n}+1}\right]_0^T$

$= \frac{1}{(2-n)k}v_0^{2-n}$

(b) $v = ((n-1)kt + v_0^{1-n})^{\frac{1}{1-n}}$

$S = \int v(t)\,dt = \frac{1}{(n-2)k}((n-1)kt + v_0^{1-n})^{\frac{n-2}{n-1}} + C$

$(t, s) = (0, 0) \rightarrow C = -\frac{1}{(n-2)k}v_0^{2-n}$

Max distance: $\frac{1}{(2-n)k}v_0^{2-n}$

(c) $n = 2$: $s(t) = \int (kt + v_0^{-1})^{-1}dt$

$= \frac{1}{k}\ln(kt + v_0^{-1})$

$t \to \infty, s \to \infty$

For $n > 2$, case is similar.

$\Rightarrow$ For $n \geq 2$, there is no limit.

1.5- 4, 6, 9

4. At time t, the population $P(t)$ of a certain city is increasing at a rate that is proportional to the number of residents in the city at that time. In January 2000, the population of the city was 10,000 and by 2005 it had risen to 20,000.

(a) What will the population of the city be at the beginning of the year 2020?

(b) In what year will the population reach one million?

(a) Doubling time is 5 years.

$P(2020) = 10{,}000 \cdot 2^{\frac{2020-2000}{5}} = 160{,}000$

(b) $\frac{1{,}000{,}000}{10{,}000} = 100 = 2^k$

$k = \log_2 100 = 6.644$

$5 \cdot k = 33.22 \approx 33 \Rightarrow$ By 2033, population will reach 1M.

6. An animal sanctuary had an initial population of 50 animals. After two years, the population was 62, while after four years it was 76. Using the logistic population model, determine the carrying capacity and the number of animals in the sanctuary after 20 years.

Logistic model: $P = \frac{CP_0}{P_0 + (C-P_0)e^{-rt}}$

$r = \frac{1}{t_1} \ln\left[\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)}\right]$

$C = \frac{P_0[P_0(P_0+P_2) - 2P_0P_2]}{P_1^2 - P_0P_2}$

$\begin{cases} P_0 = 50 \quad P_1 = 62 \quad P_2 = 76 \\ t_1 = 2 \quad t_2 = 4 \end{cases}$

$r = \frac{1}{2}\ln\left[\frac{76(62-50)}{50(76-62)}\right] = 0.1323$

$C = \frac{62[62(50+76) - 2 \cdot 50 \cdot 76]}{62^2 - 50 \cdot 76} = 298.73 \approx 299$

$P(20) = \frac{299 \cdot 50}{50 + (299-50) \cdot e^{-0.1323 \cdot 20}} = 221$

9. Consider the population model

$$\frac{dP}{dt} = r(P - T)P, \quad P(0) = P_0, \qquad (1.5.7)$$

where r, T, and P_0 are positive constants.

(a) Perform a qualitative analysis of the differential equation in the initial-value problem (1.5.7) following the steps used in the text for the logistic equation. Identify the equilibrium solutions, the isoclines, and the behavior of the slope and concavity of the solution curves.

(b) Using the information obtained in (a), sketch the slope field for the differential equation and include representative solution curves.

(c) What predictions can you make regarding the behavior of the population? Consider the cases $P_0 < T$ and $P_0 > T$. The constant T is called the **threshold level**. Based on your predictions, why is this an appropriate term to use for T?

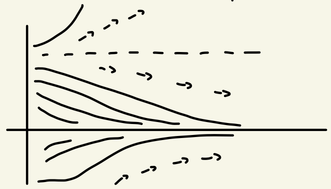
(a) Eq: $P(t) = 0$ or $P(t) = T$

Slope is positive when $P - T > 0$, or $P > T$.

Slope is negative when $P - T < 0$, or $P < T$, and $P > 0$.

Concavity is $\begin{cases} \text{up when } P > \frac{T}{2} \\ \text{down when } P < \frac{T}{2} \text{ and } P > 0 \end{cases}$

(b)



(c) Threshold level is an eq. solution

If $P_0 > T$, population increases exponentially

If $P_0 < T$, population dies out eventually

1.6 - 1, 5, 10, 24

For Problems 1–15, solve the given differential equation.

1. $\frac{dy}{dx} + y = 4e^x$.
2. $\frac{dy}{dx} + \frac{2}{x}y = 5x^2, \quad x > 0$.
3. $x^2y' - 4xy = x^7 \sin x, \quad x > 0$.
4. $y' + 2xy = 2x^3$.
5. $\frac{dy}{dx} + \frac{2x}{1-x^2}y = 4x, \quad -1 < x < 1$.
6. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4}{(1+x^2)^2}$.
7. $2(\cos^2 x)y' + y\sin 2x = 4\cos^4 x$, $0 \le x < \pi/2$.
8. $y' + \frac{1}{x\ln x}y = 9x^2$.
9. $y' - y\tan x = 8\sin^3 x$.
10. $t\frac{dx}{dt} + 2x = 4e^t, \quad t > 0$.
11. $y' = \sin x(y \sec x - 2)$.
12. $(1 - y\sin x)dx - (\cos x)dy = 0$.
13. $y' - x^{-1}y = 2x^2 \ln x$.
14. $y' + \alpha y = e^{\beta x}$, where α, β are constants.
15. $y' + mx^{-1}y = \ln x$, where m is constant.

24. Solve the differential equation for Newton's law of cooling by viewing it as a first-order linear differential equation.

1. $\mu(x) = e^{\int 1\, dx} = e^x$

$e^x y' + e^x y = 4e^{2x}$

$(e^x y)' = 4e^{2x}$

$e^x y = 2e^{2x} + C$

$y = 2e^x + Ce^{-x}$

5. $\mu(x) = e^{\int \frac{2x}{1-x^2}dx} = e^{-\ln(1-x^2)} = \frac{1}{1-x^2}$

$\frac{1}{1-x^2}y' + \frac{2x}{(1-x^2)^2}y = \frac{4x}{1-x^2}$

$\frac{1}{1-x^2}y = -2\ln(1-x^2) + C$

$y = 2(x^2-1)\ln(1-x^2) + C(1-x^2)$

10. $x' + 2\frac{x}{t} = \frac{4e^t}{t}$

$\mu(t) = e^{\int \frac{2}{t}dt} = t^2$

$t^2x' + 2tx = 4te^t$

$t^2x = 4(te^t - e^t) + C$

$x = \frac{4e^t}{t} - \frac{4e^t}{t^2} + Ct^{-2}$

24. $\frac{dT}{dt} = -k(T - T_m)$

$T' + kT = kT_m$

$\mu(t) = e^{\int k} = e^{kt}$

$(T \cdot e^{kt})' = e^{kt} \cdot k \cdot T_m$

$T \cdot e^{kt} = e^{kt} \cdot T_m + C$

$T = T_m + Ce^{-kt}$

Determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $\frac{dy}{dx} + xy^2 = 0$

2. $\frac{d^2y}{dx^2} + \cos(x+y) = e^x$

2^{nd} order differential equation nonlinear ($\cos(x+y)$)

Verify that each given function is a solution of the differential equation.

3. $ty' - y = t^2\,;\quad y = 3t + t^2$

4. $2t^2y'' + 3ty' - y = 0,\ \ t > 0\,;\quad y_1(t) = t^{\frac{1}{2}},\ y_2(t) = t^{-1}$

$Y_1(t): y' = \frac{1}{2}t^{-\frac{1}{2}},\ y'' = -\frac{1}{4}t^{-\frac{3}{2}}$

$LHS = -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0 = RHS$ ✓

$Y_2(t): y' = -t^{-2},\ y'' = 2t^{-3}$

$LHS = 4t^{-1} - 3t^{-1} - t = 0 = RHS$ ✓

Determine the value of r for which the given differential equation has a solution of the form $y = e^{rt}$.

5. $y' + 2y = 0$

6. $y''' - 3y'' + 2y' = 0$

$r^3e^{rt} - 3r^2e^{rt} + 2re^{rt} = 0$

$r(r-1)(r-2) = 0$

$r = \{0, 1, 2\}$

Determine the value of r for which the given differential equation has a solution of the form $y = t^r$ for $t > 0$.

7. $t^2y'' + 4ty' + 2y = 0$

8. $t^2y'' - 4ty' + 4y = 0$

$t^r r(r-1) - 4rt^r + 4t^r = 0$

$t^r\left[r(r-1) - 4r + 4\right] = 0$

$t^r(r-1)(r-4) = 0$

$r = \{1, 4\}$

Verify that $y(t)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(t)$ satisfies the given initial condition.

9. $y' + (\sin t)y = 0\,;\ y(t) = Ce^{\cos t},\ \ y(\pi) = 1$

10. $y' + 2y = 0;\ y(t) = Ce^{-2t},\ y(0) = 1$

$Y(t) = Ce^{-2t},\ y' = -2Ce^{-2t}$

$LHS = -2Ce^{-2t} + 2Ce^{-2t} = 0 = RHS$

$Y(0) = 1 \Rightarrow C = 1$

Solve the following problems using the Separation of variables Method.

11. $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

12. $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)},\ \ y(0) = -1$

13. $\frac{dy}{dx} = \frac{4x-x^3}{4+y^3}$

14. $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^y}$

15. $\frac{dy}{dx} = \frac{e^{-x}-e^x}{3+4y},\ \ y(0) = 1$

16. Solve the initial value problem $\frac{dy}{dx} = \frac{1+3x^2}{3y^2-6y},\ \ y(0) = 1$ and determine the interval in which the solution is valid. Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

17. Solve the initial value problem $\frac{dy}{dx} = \frac{2\cos 2x}{(3+2y)},\ \ y(0) = -1$ and determine where the solution attains its maximum value.

12. $\int 2(y-1)\,dy = \int 3x^2 + 4x + 2\,dx$

$y^2 - 2y = x^3 + 2x^2 + 2x + C$

$\Rightarrow y^2 - 2y = x^3 + 2x^2 + 2x + 3$

14. $\int y + e^y\,dy = \int x - e^{-x}\,dx$

$\frac{1}{2}y^2 + e^y = \frac{1}{2}x^2 + e^{-x} + C$

16. $\int 3y^2 - 6y\,dy = \int 1 + 3x^2\,dx$

$y^3 - 3y^2 = x + x^3 + C$

$\Rightarrow y^3 - 3y^2 = x + x^3 - 2$

$y \neq 2 \rightarrow x \neq -1$

$x \in \mathbb{R},\ x \in (-\infty, -1) \cup (-1, +\infty)$

Solve the following problems using the Integrating Factor Method.

18. $y' - 2y = 4 - t$

19. $ty' + 2y = 4t^2\ ,\ \ y(1) = 2$

20. $2y' + ty = 2\ ,\ \ \ y(0) = 1$

21. $u' + ku = kT_0 + kA\sin(\omega t)$

22. $y' + 2ty = 2te^{-t^2}$

23. $y' + y = 5\sin(2t)$

24. $ty' + 2y = \sin(t),\ \ y(\frac{\pi}{2}) = 1,\ t > 0$

25. Consider the initial value problem

$y' + \frac{1}{4}y = 3 + 2\cos(2t)\,,\quad y(0) = 0$

a) Describe the behavior of the solution for large t.

b) Determine the value of t for which the solution first intersects the line $y = 12$.

26. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation $y' + ay = be^{-\lambda t}$ has the property that $y \to 0$ as $t \to \infty$. Hint: Consider the case $a = \lambda$ and $a \neq \lambda$ separately.

18. $y' - 2y = 4 - t \quad \mu(t) = e^{\int -2\,dt} = e^{-2t}$

$e^{-2t}y' - 2ye^{-2t} = 4e^{-2t} - te^{-2t}$

$ye^{-2t} = -2e^{2t} + \frac{1}{2}te^{-2t} + \frac{t}{4}e^{-2t} + C$

$Y = -2 + \frac{1}{2}t + \frac{1}{4} + C$

$= \frac{t}{2} - \frac{7}{4} + C$

20. $2y' + ty = 2 \quad \mu(t) = e^{\int t\,dt} = e^{\frac{1}{2}t^2}$

$2e^{\frac{1}{2}t^2}y' + e^{\frac{1}{2}t^2}ty = 2e^{\frac{1}{2}t^2}$

The question cannot be represented as elementary functions

22. $y' + 2ty = 2te^{-t^2} \quad \mu(t) = e^{\int 2t\,dt} = e^{t^2}$

$e^{t^2}y = t^2 + C$

$Y = t^2e^{-t^2} + Ce^{-t^2}$

24. $y' + 2\frac{y}{t} = \frac{\sin t}{t} \quad \mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = t^2$

$(y \cdot t^2)' = t\sin t$

$y \cdot t^2 = \sin t - t\cos t + C$

$Y = \frac{\sin t}{t^2} - \frac{\cos t}{t} + Ct^{-2}$

$(\frac{\pi}{2})^{-2} + C(\frac{\pi}{2})^{-2} = 1$

$(\frac{\pi}{2})^{-2}(1 + C) = 1$

$C = (\frac{\pi}{2})^2 - 1$

$y = \frac{\sin t}{t^2} - \frac{\cos t}{t} + \left[(\frac{\pi}{2})^2 - 1\right]t^{-2}$

26. $y' + ay = be^{-\lambda t} \quad \mu(t) = e^{at}$

$a \neq \lambda \quad e^{at} \cdot y = \int be^{at - \lambda t}\,dt$

$= \frac{b}{a-\lambda}e^{(a-\lambda)t} + C$

$\Rightarrow y = \frac{b}{a-\lambda}e^{-\lambda t} + \frac{C}{e^{at}}$

$\lim_{t\to\infty} Y = \frac{b}{a-\lambda} \cdot 0 + C \cdot 0 = 0$.

$a = \lambda \quad e^{at} \cdot y = bt + C$

$y = \frac{bt + C}{e^{ac}}$

$\lim_{t\to\infty} y = 0$, by L'Hopital's rule

$\therefore \lim_{t\to\infty} y = 0$ in both cases,

$\lim_{t\to 0} y = 0\ \forall\ a \in \mathbb{R}$.