ECON 577 Homework 8

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Due November 27, 2023

Problem: Hull 15.4. Calculate the price of a 3-month European put option on a non-dividend-paying stock with a strike price of $50 when the current stock price is $50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

Sol. We follow the equation

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1).$$

Here as $S_0 = K = 50$, we see that

$$d_1 = \frac{0.1 + 0.045}{4} \cdot \frac{1}{0.3\sqrt{1/4}} \approx 0.2417, \quad d_2 = d_1 - 0.3\sqrt{1/4} \approx 0.0917.$$

Therefore

$$p \approx 50e^{-0.1/4} \times 0.4635 - 50 \times 0.4045 \approx 2.40.$$

Problem: Hull 15.5. What difference does it make to your calculations in problem 15.4 if a dividend of $1.50 is expected in 2 months?

Sol. The present value of the dividend is

$$1.5e^{-0.1\times 2/12} \approx 1.4752.$$

Therefore we could consider $S_0^* = 50 - 1.4752 = 48.5248$, and therefore

$$p \approx 3.00.$$

In essence, the option price increases by $1.4752 \times N(-d_1) \approx 0.60$.

Problem: Hull 15.8. A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is $38.

(a) What is the probability that a European call option on the stock with an exercise price of $40 and a maturity date in 6 months will be exercised?

(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

Sol. We expect $S_T = S_0 e^{xT}$, where

$$x \sim \text{Normal}(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}) \Rightarrow xT \sim \text{Normal}(T(\mu - \frac{\sigma^2}{2}), \sigma^2).$$

Considering $T = 0.5$, we have that $xT \sim N(0.049375, 0.35)$. Exercising the option requires the return to be at least $40/38 - 1 = 0.05263$, which is 0.0093 standard deviation above mean. We therefore expect the put option to have a higher probability of 50.37%, and the call option to have a lower probability of 49.63%.

Problem: Hull 15.11. Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.

(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. The risk-free rate is r.

(b) Confirm that your price satisfies the differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

Sol of part (a). At t, we have that the expected value

$$\mathbb{E}[\ln S_T] = \ln S + \left(\mu - \frac{\sigma^2}{2}\right)(T-t).$$

In the context of risk-neutral valuation, we set $\mu = r$, and therefore

$$\mathbb{E}[S_T] = \ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t) \Rightarrow S_t = e^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right].$$

Sol of part (b). We calculate as follows.

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] - e^{-r(T-t)}\left(r - \frac{\sigma^2}{2}\right),$$

$$\frac{\partial f}{\partial s} = \frac{e^{-r(T-t)}}{S} \Rightarrow \frac{\partial^2 f}{\partial s^2} = -\frac{e^{-r(T-t)}}{S^2}.$$

In turn, the BSM differential equation satisfies

$$\begin{aligned}
&\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \\
&= e^{-r(T-t)}\left[r\ln S + r\left(r - \frac{\sigma^2}{2}\right)(T-t) - \left(r - \frac{\sigma^2}{2}\right) + r - \frac{\sigma^2}{2}\right] \\
&= re^{-r(T-t)}\left[\ln S + r\left(r - \frac{\sigma^2}{2}\right)(T-t)\right] \\
&= rf.
\end{aligned}$$

Problem: Hull 15.13. What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is 3 months?

Sol. Apply the Black-Scholes equation directly. We have that

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2);$$

$$d_1 = \frac{\ln \frac{52}{50} + (0.12 + 0.045) \times 0.25}{0.30 \times 0.5} \approx 0.5365, \quad d_2 = d_1 - 0.30 \times 0.5 \approx 0.3865.$$

Therefore

$$c = 52 \times 0.7042 - 50e^{-0.12 \times 0.25} \times 0.6504 \approx 5.06.$$

Problem: Hull 15.14. What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months?

Sol. Apply the Black-Scholes equation directly. Again we have that

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1);$$

$$d_1 = \frac{\ln \frac{69}{70} + (0.05 + 0.1225/2) \times 0.5}{0.35 \times \sqrt{0.5}} \approx 0.1666, \quad d_2 = d_1 - 0.35 \times \sqrt{0.5} \approx -0.0809.$$

Therefore

$$p = 70e^{-0.05 \times 0.5} \times 0.5322 - 69 \times 0.4338 \approx 6.40.$$