Goode Linear Algebra Notes

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0 Preface

This notebook serves as notes for Stephen Goode and Scott Annin's *Differential Equations and Linear Algebra, 4th edition*. This book serves as the primary textbook for undergraduate linear algebra and differential equations, including the class that I am currently in, MATH 225. My course, as it focuses more on linear algebra than differential equations, covers a majority of chapters 1-4 and some of chapters 6,7,8, and 9. Again, if I have time, I will come back and make up for the sections not mentioned by my professor.

It is worth mentioning that my (fairly garbage) LaTeX skills does not allow me to draw any sorts of graphs or figures, so let's stick with the numbers and letters for now.

Welcome.

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1 First-Order Differential Equations

1.1 Differential Equations Everywhere

Definition 1.1: Differential equation

A **differential equation** is any equation that involves one or more derivatives of an unknown function.

Differential equations in which the unknown function depends on a single independent variable are called **ordinary differential equations**. Differential equations that involves partial derivatives of the unknown function of two independent variables are called **partial differential equations**.

The highest derivative that occurs in the differential equation is called the **order** of the differential equation.

The **Malthusian model** for the growth of a population assumes that the rate of growth is proportional to the population present at that time. The growth model can be described as

$$\frac{dP}{dt} = kP,$$

where k is a constant. It follows that

$$P(t) = Ce^{kt},$$

where C is an arbitrary constant. The above formula is called the **general solution** to the differential equation. To determine a particular solution, we also need an initial condition which specifies the appropriate value of C. For example, the initial condition $P(0) = P_0$ and $P(0) = Ce^{k\cdot 0} = C$ gives $C = P_0$, and the particular solution becomes

$$P(t) = P_0 e^{kt}.$$

The Malthusian model only predicts certain population, like bacteria. Its more general population alternative, the **logistic population model**, assumes a constant birthrate B_0 and a death rate D_0 that is proportional to the population. The resulting differential equation becomes

$$\frac{dP}{dt} = (B_0 - D_0 P)P,$$

where B_0 and D_0 are positive constants. The logistic population model has a **carrying capacity** of the population, given by $C = B_0/D_0$.

Now consider another example: the rate of change of temperature of an object. We know from thermodynamics that if the temperature of the object is hopper than that of the room, then the object will begin to cool, and vice verse. We also expect that the major factor governing the rate of cooling is the difference in temperature. The **Newton's law of cooling** arises from this. Let $T(t)$ denote the temperature of the object at time t, and let $T_m(t)$ denote the temperature of the surrounding medium. Hence,

$$\frac{dT}{dt} = -k(T - T_m) \Rightarrow T(t) = T_m + Ce^{-kt},$$

where c is a constant. Newton's law of cooling therefore predicts that as $t \to \infty$, $T \to T_m$.
Of course, there are more examples of differential equations, like the initial trajectory problem of motion, and the different types of growth models. We won't go over them one by one, but this should give a sense of how common differential equations are in our daily lives.

1.2 Basic Ideas and Terminology

We now formalize the ideas introduced in the examples in last section.
Any differential equation of order n can be written in the form

$$G(x, y, y', y'', ..., y^{(n)}) = 0,$$

where $y^{(n)}$ denotes the n-th derivative of y with respect to x.

Definition 1.2: Linear differential equations

A differential equation that can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)y = F(x),$$

where $a_0, a_1, ..., a_n$ and F are functions of x only, is called a **linear** differential equation of order n.
A differential equation that does not satisfy this definition is called a **nonlinear** differential equation.

Example 1.3. The equation $y''' + e^{3x}y'' + x^3y' + (\cos x)y = \ln x$ is a linear differential equation of order 3, whereas $y'' + y^2 = 0$ is a nonlinear differential equation of order 2.

Definition 1.4: Solution to differential equation

A function $y = f(x)$ that is n times differentiable on an interval I is called a **solution** to the differential equation on I if the substitution of $y = f(x), y' = f'(x), ...$ reduces the differential equation to an identity for all $x \in I$. In this case, we say that $y = f(x)$ **satisfies** the differential equation.

Example 1.5. We want to verify that $\sin(xy) + y^2 - x = 0$ defines a solution to

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}.$$

To do this, we apply rules of implicit differentiation.

$$\cos(xy)\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} - 1 = 0 \Rightarrow \frac{dy}{dx}[x\cos(xy) + 2y] = 1 - y\cos(xy),$$

and moving terms gives the original expression as required.

Definition 1.6: General solution

A solution to an n-th order differential equation on an interval I is called the **general solution on I** if it satisfies the following conditions:

(1) The solution contains n constants $c_1, c_2, ..., c_n$.

(2) All solutions can be obtained by assigning appropriate values to the constants.

On the other hand, a solution to a differential equation is called a **particular solution** if it does not contain any arbitrary constants not present in the differential equation itself.

Now we define the initial-value problem.

Definition 1.7: Initial-value problem

A n-th order differential equation together with n auxiliary conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \dots, y_{n-1}$ are constants, is called an **initial-value problem**.

Theorem 1.8

Let $a_1, a_2, \dots, a_n, F$ be functions that are continuous on an interval I. Then, for any $x_0 \in I$, the initial-value problem

$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

has a unique solution on I.

Proof. This is a fundamental result of differential equations, and we will prove it later in chapter 8. □

1.3 The Geometry of First-Order Differential Equations

Definition 1.9: Solution curve

The graph of any solution to the differential equation $\frac{dy}{dx} = f(x, y)$ is called a **solution curve**.

Consider an initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Geometrically, we are interested in finding the particular solution curve to the differential equation that passes through (x_0, y_0) in the Cartesian plane. The following questions naturally arise:

(1) Existence: Does the initial-value problem have any solutions?

(2) Uniqueness: If the answer to (1) is yes, does the initial-value problem have only one solution?

Theorem 1.10: Existence and Uniqueness Theorem

Let $f(x, y)$ be a function that is continuous on the rectangle

$$R = \{(x, y) : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}.$$

Suppose further that $\frac{\partial f}{\partial y}$ is continuous in R, then for any interior point (x_0, y_0) in R, there exists an interval I containing x_0 such that the initial-value problem has a unique solution for $x \in I$.

Proof. Geometrically, if $f(x,y)$ satisfies the hypotheses of the existence and uniqueness theorem in R, then throughout that region the solution curves of the differential equation $\frac{dy}{dx} = f(x,y)$ cannot intersect, or else it would imply the existence of more than one solution. $\square$

Example 1.11. The initial-value problem

$$\frac{dy}{dx} = 3xy^{\frac{1}{3}} \quad y(0) = a$$

has a unique solution whenever $a \neq 0$, but it does not have a unique solution when $a = 0$. Consider $f(x,y) = 3xy^{1/3}$, $\partial f/\partial y = xy^{-2/3}$. f is continuous in the xy-plane, but $\partial f/\partial y$ is continuous at all $y \neq 0$. Hence, it is clear that if $a \neq 0$, we can draw a rectangle containing $(0,a)$ that does not intersect the x-axis, so the Existence and Uniqueness theorem is satisfied. However, we cannot do that for $a = 0$, hence the initial-value problem have more than one solution: consider $y(x) = 0$ and $y(x) = x^3$.

Definition 1.12: Slope field

The **slope field** for a differential equation $\frac{dy}{dx} = f(x,y)$ is the sketch of the value of $f(x,y)$ at several points and drawing through each of the corresponding points in the xy-plane as their slopes.

The slope field can be sketched with three important steps:

(1) *Isoclines*: for the differential equation $\frac{dy}{dx} = f(x,y)$, the function $f(x,y)$ determines the regions in the xy-plane where the slope is positive and where it's negative. The family of curves where $f(x,y) = k$ are called **isoclines** of the differential equation.

(2) *Equilibrium solutions*: any solution to the differential equation of the form $y(x) = y_0$ where y_0 is a constant is called an **equilibrium solution** to the differential equation. The corresponding solution curve is a line parallel to the x-axis. Equilibrium solutions are given by any constant values of y for which $f(x,y) = 0$.

(3) *Concavity changes*: differentiating the differential equation with respect to x gives an expression for d^2y/dx^2 in terms of x and y, which can be useful in determining the behavior of the concavity of the solution curves.

1.4 Separable Differential Equations

Definition 1.13: Separable differential equation

A first-order differential equation is **separable** if it can be written in the form

$$p(y)\frac{dy}{dx} = q(x).$$

Theorem 1.14

If $p(y)$ and $q(x)$ are continuous, then the separable differential equation has the general solution

$$\int p(y)\,\mathrm{d}y = \int q(x)\,\mathrm{d}x + C,$$

where C is an arbitrary constant.

Proof.

$$p(y)\frac{dy}{dx} = q(x) \Rightarrow \frac{d}{dx}\left(\int p(y)\,\mathrm{d}y\right) = q(x).$$

Integration both sides of this equation with respect to x yields

$$\int p(y)\,\mathrm{d}y = \int q(x)\,\mathrm{d}x + C.$$

□

Example 1.15. We want to find all solutions to $y' = -2xy^2$. We do this by separating variable:

$$y^{-2}\mathrm{d}y = -2x\mathrm{d}x.$$

Integrating both sides gives

$$-y^{-1} = -x^2 + C \Rightarrow y(x) = \frac{1}{x^2 - C}.$$

However, we can see by inspection that $y(x) = 0$ is also a solution to the differential equation, but we did not count it in when we divided by y in separating the variables. Thus, the solutions are

$$y(x) = \frac{1}{x^2 - c} \quad \text{and} \quad y(x) = 0.$$

1.5 Some Simple Population Models

We consider in detail the models of population growth, namely Malthusian and Logistic models.
Recall that the **Malthusian growth model** is defined by the equation

$$P(t) = P_0 e^{kt},$$

where P_0 denotes the population at $t = 0$. This law predicts an exponential increase the population with time.
The time taken for population to double is the **doubling time**. This is the time, t_d, when $P(t_d) = 2P_0$. Substituting gives $2P_0 = P_0 e^{kt_d}$. Dividing both sides by P_0 and taking logarithms give

$$kt_d = \ln 2 \Rightarrow t_d = \frac{1}{k}\ln 2.$$

The general logistic model describes the rate of change of population by

$$\frac{dP}{dt} = [B(t) - D(t)]P,$$

where $B(t)$ and $D(t)$ denote the birth rate and death rate per individual, respectively. The exponential law corresponds to the case when $B(t) = k$ and $D(t) = 0$. Considering the death rate per individual is directly proportional to the population, the differential equation gives

$$\frac{dP}{dt} = (B_0 - D_0 P)P,$$

where B_0 and D_0 are positive constants. It is useful to write the differential equation in the equivalent form

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P,$$

where $r = B_0$ and $C = \frac{B_0}{D_0}$. The constant C is the **carrying capacity** of the population. If $P < C$, then $\frac{dP}{dt} > 0$ and the population increases. If $P > C$, then $\frac{dP}{dt} < 0$ and the population decreases.
Considering the analytical solution to the logistic model, we have

$$\int \frac{C}{P(C-P)}\,\mathrm{d}P = rt + C_1 \Rightarrow \ln\left|\frac{P}{C-P}\right| = rt + C_1.$$

Redefining the integration constant yields

$$\frac{P}{C-P} = C_2 e^{rt} \Rightarrow P(t) = \frac{C_2 C e^{rt}}{1 + C_2 e^{rt}} = \frac{C_2 C}{C_2 + e^{-rt}}.$$

Imposing the initial condition $P(0) = P_0$ gives $C_2 = P_0/(C - P_0)$, which gives

$$P(t) = \frac{CP_0}{P_0 + (C - P_0)e^{-rt}}.$$

1.6 First-Order Linear Differential Equations

Definition 1.16: First-order linear differential equation

A differential equation that can be written in the form

$$a(x)\frac{dy}{dx} + b(x)y = r(x),$$

where $a(x)$, $b(x)$ and $r(x)$ are functions defined on an interval (a, b), is called a **first-order linear differential equation**.

We assume that $a(x) \neq 0$ on (a, b), so that we can divide both sides by $a(x)$ to obtain the standard form

$$\frac{dy}{dx} + p(x)y = q(x).$$

The idea behind the solution is to rewrite the differential equation in the form

$$\frac{d}{dx}[g(x, y)] = F(x)$$

for an appropriate function $g(x, y)$. We multiply the function

$$I(x) = e^{\int p(x)\,\mathrm{d}x},$$

called the **integration factor** for the differential equation, as it enables us to reduce the differential equation.

Example 1.17. We want to solve the initial-value problem

$$\frac{dy}{dx} + xy = xe^{x^2/2} \quad y(0) = 1.$$

An appropriate integrating factor is $I(x) = e^{x^2/2}$, and multiplying the given differential equation by I yields

$$\frac{d}{dx}(e^{x^2/2}y) = xe^{x^2}.$$

Integrating both sides with respect to x, we obtain

$$e^{x^2/2}y = \frac{1}{2}e^{x^2} + C \Rightarrow y(x) = e^{-x^2/2}\left(\frac{1}{2}e^{x^2} + C\right).$$

Substituting the initial condition $y(0) = 1$ gives $c = \frac{1}{2}$, thus the required particular solution is

$$y(x) = \frac{1}{2}e^{-x^2/2}(e^{x^2} + 1) = \frac{1}{2}(e^{x^2/2} + e^{-x^2/2}) = \cosh(\frac{x^2}{2}).$$

2 Matrices and Systems of Linear Equations

Any equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

in constants $a_1, a_2, ..., a_n, b$ and unknowns $x_1, x_2, ..., x_n$ is called a **linear equation**. Often, several linear equations need to be considered at once, in which case we refer to a **system** of linear equations. The next two chapters are concerned with giving a introduction to matrix theory and the solution techniques for such systems.

2.1 Matrices: Definitions and Notation

Well, get ready for twenty consecutive definitions. It is a lot, but fortunately the concepts are not as difficult as other sets of concepts. Yes, I'm talking about you, metric spaces :)

Definition 2.1: Matrices

An $m \times n$ **matrix** is a rectangular array of numbers arranged in m horizontal rows and n vertical columns. Matrices are usually denoted by upper case letters, such as A and B. The entries in the matrix are called the **elements** of the matrix.

A general $m \times n$ matrix A is written as

$$A = \begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & am2 & \dots & a_{mn} \end{bmatrix}.$$

Definition 2.2: Equivalent matrices

Two matrices A and B are **equal**, written $A = B$, if they both have the same size $m \times n$, and all corresponding elements have the matrices are equal: $a_{ij} = b_{ij}$ for all i and j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Definition 2.3: Row and column vectors

A $1 \times n$ matrix is called a **row** n**-vector**. An $n \times 1$ matrix is called a **column** n**-vector**. The elements of a row or column n-vector are called the **components** of the vector.

Definition 2.4: Transpose

Interchanging the row vectors and column vectors in an $m \times n$ matrix A gives an $n \times m$ matrix, called the **transpose** of A, denoted as A^T. The ij-th element of A^T, denoted a^T_{ij}, is given by

$$a^T_{ij} = a_{ji}.$$

Definition 2.5: Square matrices

An $n \times n$ matrix is called a **square matrix**. If a is a square matrix, then the elements a_{ii}, $1 \leqslant i \leqslant n$, make up the **main diagonal** of the matrix. The sum of the main diagonal elements of an $n \times n$ matrix A is called the **trace** of A and is denoted $\mathrm{tr}(A)$.

Definition 2.6: Types of matrices

An $n \times n$ matrix $A = \left[a_{ij}\right]$ is said to be **lower triangular** if $a_{ij} = 0$ whenever $i < j$, and it is said to be **upper triangular** if $a_{ij} = 0$ whenever $i > j$. An $n \times n$ matrix $D = \left[d_{ij}\right]$ is said to be a **diagonal matrix** if $d_{ij} = 0$ whenever $i \neq j$. Note that a matrix D is a diagonal matrix if and only if D is simultaneously upper and lower triangular, and it can be represented in the compact form

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_n).$$

Definition 2.7: Symmetric matrix

A square matrix A satisfying $A^T = A$ is called a **symmetric matrix.**
If $A = \left[a_{ij}\right]$, then we let $-A$ denote the matrix with elements $-a_{ij}$. A square matrix A satisfying $A^T = -A$ is called a **skew-symmetric** (or **anti-symmetric**) **matrix**.

Definition 2.8: Matrix functions

An $m \times n$ **matrix function** A is a rectangular array with m rows and n columns whose elements are functions of a single real variable t. The matrix function is only defined for real values of t such that all elements in $A(t)$ assume a well-defined value.

An $n \times 1$ matrix function is called a **column n-vector function**.

2.2 Matrix Algebra

After having a general idea of a matrix, the next step is to develop the algebra of matrices. We assume that all elements of the matrices are real or complex numbers.

Definition 2.9: Addition

If A and B are both $m \times n$ matrices, we define **addition**, or the **sum**, of A and B, denoted by $A + B$, to be the $m \times n$ matrix $A + B = \left[a_{ij} + b_{ij}\right]$.

Matrix addition is commutative and associative.

Definition 2.10: Scalar multiplication

If A is an $m \times n$ matrix and s is a scalar, we define **scalar multiplication** of s and A, denoted by sA, be the $m \times n$ matrix $sA = \begin{bmatrix} sa_{ij} \end{bmatrix}$.

Scalar multiplication is associative and distributive, both over matrices and over scalars.

Definition 2.11: Subtraction

If A and B are both $m \times n$ matrices, we define **subtraction** of A and B, denoted $A - B$, be the $m \times n$ matrix $A - B = A + (-1)B = \begin{bmatrix} a_{ij} - b_{ij} \end{bmatrix}$.

The general definition of matrix multiplication can be built up in three stages. Matrix multiplication may seem unapparent at first glance, but later in section 6.5, the application of matrix multiplication in linear transformations will be more transparent. Get ready, and we are going to accelerate.

Case 1: product of a row n-vector and a column n-vector. Let a be a row n-vector, and let x be a column n-vector. Their matrix product ax is the 1×1 matrix whose single element is obtained by taking the dot product of a and x^T. Mathematically,

$$ax = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 x_1 + x_2 x_2 + \dots + a_n x_n \end{bmatrix}.$$

Example 2.12. If $a = \begin{bmatrix} -8 & 3 & 1 & 2 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 4 \\ 7 \\ -5 \end{bmatrix}$, then

$$ax = \begin{bmatrix} (-8)(1) + (3)(4) + (1)(7) + (2)(-5) \end{bmatrix} = \begin{bmatrix} -13 \end{bmatrix}.$$

Case 2: product of an $m \times n$ matrix and a column n-vector. If A is an $m \times n$ matrix and x is a column n-vector, then the product Ax is defined to be the $m \times 1$ matrix whose i-th element is obtained by taking the dot product of the i-th row vector of A with x. Mathematically,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} (Ax)_1 \\ (Ax)_2 \\ \vdots \\ (Ax)_i \\ \vdots \\ (Ax)_m \end{bmatrix},$$

and Ax has i-th element

$$(Ax)_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n.$$

Example 2.13. If $A = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 5 & -2 \\ -4 & -2 & 5 \end{bmatrix}$ and $x = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$, then

$$Ax = \begin{bmatrix} (-3)(-2) + (1)(1) + (-2)(6) \\ (0)(-2) + (5)(1) + (-2)(6) \\ (-4)(-2) + (-2)(1) + (5)(6) \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 36 \end{bmatrix}.$$

Before we get to case 3, it is helpful to introduce a theorem that is closely related to case 2.

Theorem 2.14

If $A = \begin{bmatrix} a_1, a_2, \ldots, a_n \end{bmatrix}$ is an $m \times n$ matrix and $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a column n-vector, then

$$Ac = c_1a_1 + c_2a_2 + \ldots + c_na_n.$$

Proof. Define $(x)_i$ as the i-th element of x. The element a_{ik} of A is the i-th component of the column m-vector a_k, so

$$a_{ik} = (a_k)_i.$$

Hence, applying the results for case 2 gives

$$(Ac)_i = \sum_{k=1}^{n} a_{ik}c_k = \sum_{k=1}^{n} (a_k)_i c_k = \sum_{k=1}^{n} (c_k a_k)_i.$$

Consequently,

$$Ac = \sum_{k=1}^{n} c_k a_k = c_1a_1 + c_2a_2 + \ldots + c_na_n.$$

□

If $a_1, a_2, \ldots, a_n$ are column m-vectors and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1a_1 + c_2a_2 + \ldots + c_na_n$$

is called a **linear combination** of the column vectors.

Case 3: product of an $m \times n$ matrix and an $n \times p$ matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB has columns defined by multiplying the matrix A by the respective column vectors of B. That is, if $B = \begin{bmatrix} b_1, b_2, \ldots, b_p \end{bmatrix}$, then AB is the $m \times p$ matrix defined by

$$AB = \begin{bmatrix} Ab_1, Ab_2, \ldots, Ab_p \end{bmatrix}.$$

Example 2.15. If $A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 \\ 3 & -1 \\ -9 & 2 \end{bmatrix}$, then

$$AB = \begin{bmatrix} (-2)(-4)+(1)(3)+(3)(-9) & (-2)(1)+(1)(-1)+(3)(-2) \\ (4)(-4)+(-2)(3)+(6)(-9) & (4)(1)+(-2)(-1)+(6)(2) \end{bmatrix} = \begin{bmatrix} -16 & 3 \\ -76 & 18 \end{bmatrix}.$$

Definition 2.16: Index form of matrix product

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times n$ matrix, $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ is an $n \times p$ matrix, and $C = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant p.$$

This is the **index form** of the matrix product.

In order for the product AB to be defined, A and B must satisfy

$$\# \text{ columns of } A = \# \text{ rows of } B.$$

Theorem 2.17

If A, B and C have appropriate dimensions for the operations to be performed, then matrix multiplication is associative, left distributive, and right distributive. That is,

$$A(BC) = (AB)C \quad A(B+C) = AB + AC \quad (A+B)C = AC + BC.$$

Proof. Associative property is trivial. Consider the right distributive property,

$$\begin{aligned} \left[(A+B)C\right]_{ij} &= \sum_{k=1}^{n} (a_{ik} + b_{ik}) c_{kj} = \sum_{k=1}^{n} a_{ik} c_{kj} + \sum_{k=1}^{n} b_{ik} c_{kj} \\ &= (AC)_{ij} + (BC)_{ij} \\ &= (AC + BC)_{ij}. \end{aligned}$$

It follows that $(A+B)C = AC + BC$. The left distributive property can be proven by similar manners. □

It is worth noting that except for rare, special cases, matrix multiplication is not commutative. That is,

$$AB \neq BA.$$

It is easy to find counterexamples: consider A as a $m \times n$ matrix, and B as a $n \times m$ matrix. AB gives a $m \times m$ matrix, whereas BA gives a $n \times n$ matrix. If $m \neq n$, then $AB \neq BA$ by the definition of equivalent matrices.

For an $n \times n$ matrix, we use the usual power notation to denote the operation of multiplying A by itself. That is, $A^2 = AA$, $A^3 = AAA$. However, powers of matrices are not used as often in elementary linear algebra.

Definition 2.18: Identity matrix

The **identity matrix**, I_n, is the $n \times n$ matrix with ones on the main diagonal and zeros elsewhere. The elements of I_n can be represented by the **Kronecker delta symbol**, δ_{ij}, defined by

$$\delta_{ij} = 1 \quad i = j, \quad \delta_{ij} = 0 \quad i \neq j.$$

Then,

$$I_n = \left[\delta_{ij}\right].$$

The identity matrix have an important property, that it plays the same role in matrix multiplication as the number 1 does in the multiplication of real numbers. Namely,

$$A_{m\times n} I_n = A_{m\times n} \quad I_m A_{m\times p} = A_{m\times p}.$$

We now discuss the properties of the transpose.

Theorem 2.19

Let A and C be $m \times n$ matrices, and let B be an $n \times p$ matrix. Then

(1) $(A^T)^T = A$.

(2) $(A + C)^T = A^T + C^T$.

(3) $(AB)^T = B^T A^T$.

Proof. We prove (3) here from the definition of the transpose and the index form. Statements (1) and (2) are simple enough (I believe).

$$\begin{aligned}\left[(AB)^T\right]_{ij} &= (AB)_{ji} \\ &= \sum_{k=1}^{n} a_{jk} b_{ki} \\ &= \sum_{k=1}^{n} b_{ki} a_{jk} = \sum_{k=1}^{n} b^T_{ik} a^T_{kj} \\ &= (B^T A^T)_{ij}.\end{aligned}$$

Consequently, $(AB)^T = B^T A^T$. $\square$

Here we skip the proof for triangular matrices and matrix functions. For triangular matrices, it is worth knowing that the product of two lower triangular matrices is a lower triangular matrix, and the product of two upper triangular matrices is an upper triangular matrix. For matrix functions, the algebra is the same as algebra for matrices, and the derivatives and integrals are simply the derivative and integral for every single element. Note that product rule applies here, that is,

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B.$$

2.3 Terminology for Systems of Linear Equations

Definition 2.20: System of equations

The general $m \times n$ system of linear equations is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

where the **system coefficients** a_{ij} and the **system constants** b_j are given scalars and $x_1, x_2, \ldots, x_n$ denote the unknowns in the system. If $b_i = 0$ for all i, then the system is called **homogeneous**; otherwise it is called **nonhomogeneous**.

By a **solution** to the system, we mean an ordered n-type of scalars $(c_1, c_2, \ldots, c_n)$, which, when substituted for $x_1, x_2, \ldots, x_n$ into the left-hand side of the system, yield the values on the right-hand side. The set of all solutions to the system is called the **solution set** to the system.

A system of equations that has at least one solution is said to be **consistent**, whereas a system that has no solution is called **inconsistent**.

Our problem will be to determine whether a given system is consistent and then , in the case when it is, to find its solution set. To do this, we first define the matrix of coefficients and the augmented matrix.

Definition 2.21: A and $A^{\#}$

Naturally associated with the system of linear equations are the following two matrices:

(1) The **matrix of coefficients** $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$.

(2) The **augmented matrix** $A^{\#} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$.

The augmented matrix $A^{\#}$ completely characterizes a system of equations since it contains all the system coefficients and the system constants. We will see in the following sections that the relationship between A and $A^{\#}$ determines the solution properties of a linear system.

The $m \times n$ general system of linear equations can be written as the **vector equation**

$$Ax = b,$$

where A is the $m \times n$ matrix of coefficients,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ v_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The column n-vector x is the **vector of unknowns**, and the column m-vector b is the **right-hand side vector**.

2.4 Row-Echelon Matrices and Elementary Row Operations

Consider the system of equations

$$\begin{aligned} x_1 + a_2 x_2 + a_3 x_3 &= y_1 \\ x_2 + b_3 x_3 &= y_2 \\ x_3 &= y_3 \end{aligned}$$

This system can be solved quite easily. From $x_3 = y_3$ in the third equation, we substitute back to the second equation to obtain x_2, and we substitute back to the first equation to obtain x_1. This technique is called **back substitution**. This section deals with characteristics of a linear system that can be solved by back substitution quite easily.

First, we define a row-echelon matrix.

Definition 2.22: Row-echelon matrix

An $m \times n$ matrix is called a **row-echelon matrix** if it satisfies the following three conditions:

(1) If there are any rows consisting entirely of zeros, they are grouped together at the bottom of the matrix.

(2) The first nonzero element in any nonzero row is a 1. (It is called a **leading 1**.)

(3) The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

Example 2.23. $\begin{bmatrix} 1 & -8 & -3 & 7 \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a row-echelon matrix, whereas $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ isn't.

Note that not all matrices are row-echelon matrices. However, every matrix A can be reduced to row-echelon form in a way that the solution set of the linear equation system with the coefficient matrix A remains unaltered. In general, the following three operations can be performed on any $m \times n$ system of linear equations without altering the solution set:

(1) Permute equations.

(2) Multiply an equation by a nonzero constant.

(3) Add a multiple of one equation to another equation.

A similar statement can be made for the augmented matrix of the system. These operations are called **elementary row operations** and is essential even for matrices not derived from linear equation systems. Consider the following notations, all of which are going to be very important:

(1) P_{ij} : permute the i-th and j-th rows of A.

(2) $M_i k$: multiply every element of the i-th row of A by a nonzero scalar k.

(3) $A_{ij}(k)$: add to the elements of the j-th row of A the scalar k times the elements of the i-th row of A.

Furthermore, the notation $A \sim B$ means that matrix B has been obtained from matrix A by a sequence of elementary row operation.

Definition 2.24: Row equivalency

Let A be an $m \times n$ matrix. Any matrix obtained from A by a finite sequence of elementary row operations is said to be **row-equivalent** to A.

Example 2.25. We want to reduce $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$ to row-echelon form.

We do this step by step. We first put a leading 1 in the (1,1) position by P_{12}.

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

Then, we use the leading 1 in the (1,1) position to clean up the first column, because essentially there should be no non-zero elements under each leading 1. We do $A_{12}(-2), A_{13}(4), A_{14}(-2)$ to get

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

Now we put a leading 1 in the (2,2) position. We cannot do it by permutation now, so let's consider $A_{32}(-1)$.

After, we use the new leading 1 to clean up the second column, namely $A_{23}(-2), A_{24}(-2)$.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 9 & 9 \end{bmatrix}$$

Now we put a leading 1 in the (3,3) position. Doing it by permutation and addition isn't as convenient as multiplication, so we do $M_3(1/13)$. We then use the leading 1 to clean up the third column with $A_{34}(-9)$.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 9 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then we try to put a leading 1 in the (4,4) position, but note that the matrix above is already in row-echelon form, so we are good by just leaving it there. Also note that for each step that results in a leading 1, we used permutation for (1,1), addition for (2,2), and multiplication for (3,3). In fact, we can use any of the three at any time, but sometimes one method is just simpler than other. For example, using $A_{43}(-4/3)$ gives us the leading 1 in (3,3), but it is far more complicated than $M_3(1/13)$.

Now, we derive some further results on row-echelon matrices that is crucial for solving systems of linear equations. First note that a row-echelon form for a matrix A is not unique. Given one row-echelon form for A, we can always obtain a different row-echelon form for A by taking the first row-echelon form for A and adding some multiple of a given row to any rows above it. However, the row-equivalent matrices shave the same number of nonzero rows. That is, they have the same **rank**.

Definition 2.26: Rank

The number of nonzero rows in any row-echelon form of a matrix A is called the **rank** of A and is denoted rank(A).

Example 2.27. We want to determine rank(A) if $A = \begin{bmatrix} 3 & -1 & 4 & 2 \\ 1 & -1 & 2 & 3 \\ 7 & -1 & 8 & 0 \end{bmatrix}$. We compute the rank by first reducing A to row-echelon form,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 3 & -1 & 4 & 2 \\ 7 & -1 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & -7 \\ 0 & 6 & -6 & -21 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -7/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are two nonzero rows in the row-echelon form of A, it follows that rank$(A) = 2$.

Here we define a special type of row-echelon matrices, called reduced row-echelon matrices.

Definition 2.28: Reduced row-echelon matrix

An $m \times n$ matrix is called a **reduced row-echelon matrix** if it is a row-echelon matrix and any column that contains a leading 1 has zeros everywhere else.

Reduced row-echelon matrices are different than normal row-echelon matrices, that an $m \times n$ matrix is row-equivalent to a *unique* reduced row-echelon matrix.

Example 2.29. We want a reduced row-echelon form of $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$. Having computed its row-echelon form $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we want to reduce the second and third column. We try apply $A_{21}(1)$ to obtain $\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then apply $A_{31}(4), A_{32}(6)$ to obtain $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, finishing the reduction.

2.5 Gaussian Elimination

We now illustrate how elementary row operations applied to the augmented matrix can be used to determine whether the system is consistent, and if the system is consistent, to find all of its solutions.

Definition 2.30: Gaussian elimination

The process of reducing the augmented matrix $A^{\#}$ to row-echelon form and then using back substitution to solve the equivalent system is called **Gaussian elimination**. The particular case of Gaussian elimination that arises when the augmented matrix is reduced to reduced row-echelon form is called **Gauss-Jordan elimination**.

Example 2.31. We want to use Gauss-Jordan elimination to determine the solution set to

$$\begin{aligned} x_1 - x_2 - 5x_3 &= -3 \\ 3x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 - 5x_3 &= 1 \end{aligned}$$

To do this, we first reduce the augmented matrix of the system to reduced row-echelon form, as follows:

$$\begin{bmatrix} 1 & -1 & -5 & -3 \\ 3 & 2 & -3 & 5 \\ 2 & 0 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -5 & -3 \\ 0 & 5 & 12 & 14 \\ 0 & 2 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -5 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -5 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 32 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{bmatrix}.$$

The augmented matrix is now in reduced row-echelon form. The equivalent system is

$$x_1 = 18$$
$$x_2 = -14$$
$$x_3 = 7$$

The solution can be read off directly as $(18, -14, 7)$.

Now we try to derive the theory for the solution sets of system of equations. Theoretical alert (proof omittable).

Theorem 2.32: Solution sets of system of equations

Lemma 2.32.1

Consider the $m \times n$ linear system $Ax = b$. Let $A^{\#}$ denote the augmented matrix of the system. If $\text{rank}(A) = \text{rank}(A^{\#}) = n$, then the system has a unique solution.

Proof. If $\text{rank}(A) = \text{rank}(A^{\#}) = n$, then there are n leading ones in any row-echelon form of A, and hence, back substitution gives a unique solution. Note that $\text{rank}(A) \leqslant \text{rank}(A^{\#})$, thus, there are only two possibilities: $\text{rank}(A) < \text{rank}(A^{\#})$ or $\text{rank}(A) = \text{rank}(A^{\#})$. □

Lemma 2.32.2

Consider the $m \times n$ linear system $Ax = b$. Let $A^{\#}$ denote the augmented matrix of the system. If $\text{rank}(A) < \text{rank}(A^{\#})$, the system is inconsistent.

Proof. If $\text{rank}(A) < \text{rank}(A^{\#})$, then there will be one row in the reduced row-echelon form of the augmented matrix whose first nonzero element arises in the last column. Such a row corresponds to an equation of the form

$$0x_1 + 0x_2 + \ldots + 0x_n = 1,$$

which has no solution. Consequently, the system is inconsistent. □

Lemma 2.32.3

Consider the $m \times n$ linear system $Ax = b$. Let $A^{\#}$ denote the augmented matrix of the system and let $r^{\#} = \text{rank}(A^{\#})$. If $r^{\#} = \text{rank}(A) < n$, then the system has an infinite number of solution, indexed by $n - r^{\#}$ free variables.

Proof. Any row-echelon equivalent system have only $r^{\#}$ equations involving the n variables, so there will be $n - r^{\#} > 0$ free variables. Assigning arbitrary values to these variables results in the uniqueness of the remaining $r^{\#}$ variables from back substitutions. Since the free variables can each assume infinitely many values, there are an infinite number of solutions to the system. □

Now we can formalize the theorem regarding the theory of linear equation systems. Consider the $m \times n$ linear system $Ax = b$. Let r denote the rank of A, and let $r^{\#}$ denote the rank of the augmented matrix of the system. Then,

(1) If $r < r^{\#}$, the system is inconsistent;

(2) if $r = r^{\#}$, the system is consistent, and

   (a) There exists a unique solution if and only if $r^{\#} = n$, and

   (b) there exists an infinite number of solutions if and only if $r^{\#} < n$.

Now we turn our attention to homogeneous linear systems. Trivially, we see that the homogeneous linear system $Ax = 0$ has at least one solution: $x = 0$. In fact, it is sometimes referred to as the **trivial solution**. Hence, we can conclude that all homogeneous linear systems are consistent for any coefficient matrix A.

However, a homogeneous system can have more than one solution. Particularly, a homogeneous system of m linear equations in n unknowns, with $m < n$, has an infinite number of solutions. This can be shown by the fact that $r = r^{\#} \leqslant m < n$ for a homogeneous systems.

Example 2.33. We want to determine the solution set to $Ax = 0$ if $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & 7 \end{bmatrix}$. Note that the first column is zero, so we set up the augmented matrix

$$A^{\#} = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equivalent system is $x_2 = 0$, $x_3 = 0$. Since x_1 does not occur, it is a free variable and the solution set to the system is therefore $S = \{(t, 0, 0) : t \in \mathbb{R}\}$.

2.6 The Inverse of a Square Matrix

Consider the situation when, for a given $n \times n$ matrix A, there exists a matrix B satisfying

$$AB = I_n \quad BA = I_n.$$

And yes, such B does exist, and it is the **inverse** of A.

Theorem 2.34: Uniqueness of inverse

Let A be an $n \times n$ matrix. Suppose B and C are both $n \times n$ matrices satisfying

$$AB = BA = I_n \quad AC = CA = I_n.$$

Then, $B = C$.

Proof. Consider $C = CI_n = C(AB)$. It follows that

$$C = C(AB) = (CA)B = I_nB = B.$$

□

Definition 2.35: Invertible matrices

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n,$$

then we call A^{-1} the matrix **inverse** to A. We also say that A is invertible if A^{-1} exists. Invertible matrices are sometimes called **nonsingular**, while noninvertible matrices are sometimes called **singular**.

Theorem 2.36

If A^{-1} exists, then the $n \times n$ system of linear equations

$$Ax = b$$

has the unique solution

$$x = A^{-1}b$$

for every $b \in \mathbb{R}^n$.

Proof. We can verify by direct substitution that $x = A^{-1}b$ is a solution to the linear system. Regarding the uniqueness of the solution, observe that for any solution x_1 to the system $Ax = b$,

$$Ax_1 = b \Rightarrow x_1 = A^{-1}b.$$

□

Theorem 2.37

An $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$.

Proof. If A^{-1} exists, then any $n \times n$ linear system $Ax = b$ has a unique solution. Hence, it is implied that $\text{rank}(A) = n$. Conversely, suppose $\text{rank}(A) = n$. Consider $e_1, e_2, \ldots, e_n$ as the column vectors of I_n. Since $\text{rank}(A) = n$, each of the linear systems

$$Ax_i = e_i$$

has a unique solution x_i. Consequently, letting $X = \begin{bmatrix} x_1, x_2, \ldots, x_n \end{bmatrix}$, where $x_1, x_2, \ldots, x_n$ are the unique solutions of $Ax_i = e_i$, gives the following equality:

$$A\begin{bmatrix} x_1, x_2, \ldots, x_n \end{bmatrix} = \begin{bmatrix} Ax_1, Ax_2, \ldots, Ax_n \end{bmatrix} = \begin{bmatrix} e_1, e_2, \ldots, e_n \end{bmatrix} \Rightarrow AX = I_n.$$

We claim that $XA = I_n$. That is,

$$(AX)A = A \Rightarrow A(XA - I_n) = 0_n.$$

We must also show that $XA - I_n = 0_n$. Let $y_1, y_2, \ldots, y_n$ denote the column vectors of the $n \times n$ matrix $XA - I_n$. Equating corresponding column vector on $A(XA - I_n) = 0_n$ implies

$$Ay_i = 0.$$

By assumption, $\text{rank}(A) = n$, so each of the systems has only the trivial solution. Consequently, each y_i is the zero vector, implying $XA - I_n = 0_n$. Therefore,

$$XA = I_n.$$

Having shown $AX = I_n$ and $XA = I_n$, we can now conclude, by definition, that $X = A^{-1}$. □

Corollary 2.38

Let A be an $n \times n$ matrix. If $Ax = b$ has a unique solution for some column n-vector b, then A^{-1} exists.

Proof. If $Ax = b$ has a unique solution, then $\text{rank}(A) = n$, hence A^{-1} exists. □

An effective method of finding A^{-1} is the **Gauss-Jordan technique**. The proof of the method is omitted here. Essentially, to find the inverse of A, create the $n \times 2n$ matrix $\begin{bmatrix} A & I_n \end{bmatrix}$ and reduce A to I_n using elementary row operations. Schematically,

$$\begin{bmatrix} A & I_n \end{bmatrix} \sim \ldots \sim \begin{bmatrix} I_n & A^{-1} \end{bmatrix}.$$

Example 2.39. We want to find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix}$. We do this through the Gauss-Jordan technique:

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -10 & -3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -14 & -3 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{14} & \frac{1}{7} & -\frac{1}{14} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{11}{14} & -\frac{8}{7} & \frac{1}{14} \\ 0 & 1 & 0 & -\frac{3}{7} & \frac{5}{7} & \frac{1}{7} \\ 0 & 0 & 1 & \frac{3}{14} & \frac{1}{7} & -\frac{1}{14} \end{bmatrix}.$$

Consequently,

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 11 & -16 & 1 \\ -6 & 10 & 2 \\ 3 & 2 & -1 \end{bmatrix}.$$

Example 2.40. Continuing the previous example, we can use A^{-1} to solve the system

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ x_2 + 2x_3 &= 1 \\ 3x_1 + 5x_2 - x_3 &= 4 \end{aligned}$$

Consider the system as $Ax = b$, where $b = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$. Since A is invertible, the system has a unique solution $x = A^{-1}b$. Thus, we have, from the previous example, that

$$x = \frac{1}{14}\begin{bmatrix} 11 & -16 & 1 \\ -6 & 10 & 2 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \\ \frac{2}{7} \end{bmatrix}.$$

Consequently, $x_1, x_2, x_3 = (5/7, 3/7, 2/7)$.

We now present some properties of the inverse.

Theorem 2.41

Let A and B be invertible $n \times n$ matrices. Then

(1) A^{-1} is invertible, and $(A^{-1})^{-1} = A$.

(2) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

(3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof. For (1), we have $A^{-1}A = AA^{-1} = I_n$ from the definition.
For (2), $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I_n$, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I_n$.
For (3), $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$, $(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$. □

2.7 Elementary Matrices and the LU Factorization

Definition 2.42: Elementary matrices

Any matrix obtained by performing a single elementary row operation on the identity matrix is called an **elementary matrix**.

In particular, an elementary matrix is always a square matrix. There are three types of matrices, corresponding to the three types of elementary row operations: P_{ij}, M_ik, $A_{ij}(k)$.

Example 2.43. In this example, we write all 2×2 elementary matrices.

(1) Permutation matrix: $P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(2) Scaling matrices: $M_1(k) = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $M_2(k) = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.

(3) Row combinations: $A_{12}(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $A_{21}(k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

More generally, the $n \times n$ elementary matrices have the following structure:
P_{ij} have ones along the main diagonal except for (i,i) and (j,j), ones in the (i,j) and (j,i), and zeros elsewhere.
$M_i(k)$ is the diagonal matrix $\text{diag}(1,1,\ldots,k,\ldots,1)$, where k appears in the (i,i) position.
$A_{ij}(k)$ have ones along the main diagonal, k in the (j,i) position, and zeros elsewhere.
It is important to note that premultiplying an $n \times p$ matrix A by an $n \times n$ elementary matrix E has the effect of performing the corresponding elementary row operation on A.

Example 2.44. We want to determine the elementary matrices that reduce $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ to row-echelon form. To do so, we can reduce A to row-echelon form with

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

The row operations used here is $P_{12}, A_{12}(-2), M_2(-1/5)$. Consequently,

$$M_2(-1/5)A_{12}(-2)P_{12}(A) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

Since elementary row operation is reversible, it follows that each elementary matrix is also invertible. We have:

$$M_i(k)^{-1} = M_i(1/k) \quad P_{ij}^{-1} = P_{ij} \quad A_{ij}(k)^{-1} = A_{ij}(-k).$$

Considering the invertible $n \times n$ matrix A. We have the following transformation:

$$E_k E_{k-1} ... E_2 E_1 A = I_n \Rightarrow A^{-1} = E_k E_{k-1} ... E_2 E_1$$

$$A = (A^{-1})^{-1} = E_1^{-1} E_2^{-1} ... E_k^{-1}.$$

Now we discuss the LU decomposition. L stands for lower triangular, and U stands for upper triangular. We start with an example.

Example 2.45. We want to use elementary row operations to reduce $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$ to upper triangular form. Here we only use the type 3 elementary row operations, as follows:

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 2 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & \frac{9}{2} & \frac{5}{2} \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & 2 \end{bmatrix}.$$

Here, we used $A_{12}(-3/2), A_{13}(1/2), A_{23}(9/13)$.

When using elementary row operations of type 3, the multiple of a specific row that is subtracted from row i to put a zero in the (i,j) position is called a **multiplier**, denoted m_{ij}. Therefore, in the preceding example, the three multipliers are

$$m_{21} = 3/2, \quad m_{31} = -1/2, \quad m_{32} = -9/13.$$

Not all matrices can be reduced to upper triangular form using only row operations of type 3. However, we restrict our attention to invertible matrices A for which the reduction to upper triangular form can be accomplished only by row operations of type 3. In terms of elementary matrices, we have

$$E_k E_{k-1} ... E_2 E_1 A = U,$$

where $E_k, E_{k-1}, ..., E_2, E_1$ are lower triangular type 3 elementary matrices and U is an upper triangular matrix. We then have

$$A = E_1^{-1}E_2^{-1}...E_k^{-1}U = LU.$$

Here, $L = E_1^{-1}E_2^{-1}...E_k^{-1}$. As all E_i are lower triangular, L is also lower triangular. Furthermore, this LU factorization is unique.

Example 2.46. We want to determine the LU factorization of $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$. Having already computed $U = \begin{bmatrix} 2 & 5 & 3 \\ 0 & -\frac{13}{2} & -\frac{13}{2} \\ 0 & 0 & -2 \end{bmatrix}$, we have

$$L = E_1^{-1}E_2^{-1}E_3^{-1},$$

hence the multipliers become useful here. We have

$$E_1^{-1} = A_{12}(\frac{3}{2)} \quad E_2^{-1} = A_{13}(-1/2) \quad E_3^{-1} = A_{23}(-9/13).$$

Substituting these results gives

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{9}{13} & 1 \end{bmatrix}.$$

The LU decomposition can be used to solve the $n \times n$ system of linear equation $Ax = b$. Consider $A = LU$, the system becomes

$$LUx = b.$$

We can then separate the system into two systems:

$$Ly = b \quad Ux = y.$$

The first system can be solved by forward substitution to get y, and the second system can be solved by backward substitution to get x.

The LU decomposition is not mentioned or used in other sections of the book (plus solving linear systems with LU is generally slower than using Gaussian elimination), so we will skip its example.

2.8 Invertible Matrix Theorem I

We now present perhaps the most important theorem of this book: the baby Invertible Matrix Theorem. Although it has six statements, it is still the "baby" version of the invertible matrix theorem, which will be discussed in section 4.10.

Theorem 2.47: Invertible matrix theorem I

Let A be an $n \times n$ matrix. The following conditions on A are equivalent:

(a) A is invertible.

(b) The equation $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

(c) The equation $Ax = 0$ has only the trivial solution $x = 0$.

(d) rank$(A) = n$.

(e) A can be expressed as a product of elementary matrices.

(f) A is row-equivalent to I_n.

Proof. The equivalence of (a)(b)(d) has been established in section 2.6. The equivalence of (a)(e) has been established in section 2.7.
Now we establish (b) implies (c) implies (d).
Assuming that (b) holds, we can conclude that the linear system $Ax = 0$ has a unique solution: the trivial solution $x = 0$. Hence, this is the unique solution, implying (c).
Assume that (c) holds, $Ax = 0$ has one trivial solution implies that reducing A to row-echelon form gives no free variables. Thus, every column and every row of A contains a pivot, meaning that the row-echelon form of A has n nonzero rows. That is, rank$(A) = n$, implying (d).
Now we establish (e) implies (f) implies (a).
Assuming that (e) holds, we can multiply I_n by a product of elementary matrices to obtain A, meaning that A is row-equivalent of I_n, implying (f).
Assuming that (f) holds, A is row-equivalent to I_n. Then, we can write A as a product of elementary matrices, each of which is invertible. Since a product of invertible matrices is invertible, we conclude that A is invertible, proving (a). □

3 Determinants

The determinant is a number, associated with an $n \times n$ matrix A, whose value characterizes when the linear system $Ax = b$ has a unique solution.

3.1 The Definition of the Determinant

We begin with the special cases $n = 1$, $n = 2$, and $n = 3$.
Case 1: $n = 1$. For a 1×1 matrix $A = \begin{bmatrix} a_{11} \end{bmatrix}$,

$$\det(A) = a_{11}$$

The matrix A is invertible if and only if $\text{rank}(A) = 1$ and inly if $\det(A)$ is nonzero.
Case 2: $n = 2$. The 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible if and only if $\text{rank}(A) = 2$, if and only if the row-echelon form of A has two nonzero rows. Given that $a_{11} \neq 0$, we can reduce A as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{bmatrix}.$$

For A to be invertible, $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Thus, it is necessary that the 2×2 determinant, $\det(A)$, defined by

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

is nonzero.
Case 3: $n = 3$. The 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is invertible if and only if $\text{rank}(A) = 3$. Reducing A to row-echelon form as in case 2, it is necessary for the 3×3 determinant defined by

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

is nonzero.

To generalize for $n \times n$ matrix A, we inspect each case in terms of their structure. Each determinant consists of a sum of $n!$ products, where each product term contains one element from each row and each column of A. Furthermore, each possible choice of one element from each row and each column occur as a term of the summation. Each term is assigned a plus or a minus sign. To tackle this, we introduce the concept of permutation. Computing determinants based off permutation isn't my favorite method, so I will only touch on it.

Definition 3.1: Permutation

Consider the first n positive integers $1, 2, 3, \dots, n$. Any arrangement of these integers in a specific order is called a **permutation**. There are $n!$ distinct permutations of the integers $1, 2, \dots, n$.

The pair of elements p_i and p_k in the permutation $(p_1, p_2, \dots, p_n)$ are said to be **inverted** if they are out of their natural order. That is, if $p_i > p_k$ with $i < j$. We say that (p_i, p_j) is an inversion. Denote $N(p_1, p_2, \dots, p_n)$ as the total number of inversions in the permutation $(p_1, p_2, \dots, p_n)$.

Definition 3.2: Parity

If $N(p_1, p_2, ..., p_n)$ is an even integer, then the permutation is even, and $(p_1, p_2, ..., p_n)$ has even parity.
If $N(p_1, p_2, ..., p_n)$ is an odd integer, then the permutation is odd, and $(p_1, p_2, ..., p_n)$ has odd parity.
Denote $\sigma(p_1, p_2, ..., p_n) = (-1)^{N(p_1, p_2, ..., p_n)}$, $\sigma = +1$ if the permutation is even, $\sigma = -1$ if it is odd.

Definition 3.3: Determinant

Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. The **determinant of** A, denoted $\det(A)$, is defined as follows:

$$\det(A) = \sum \sigma(p_1, p_2, ..., p_n) a_{1_{p_1}} a_{2_{p_2}} a_{3_{p_3}} ... a_{n_{p_n}},$$

where the summation is over the $n!$ distinct permutations $(p_1, p_2, ..., p_n)$ of the integers $1, 2, 3, ..., n$. The determinant of an $n \times n$ matrix is said to have **order** n.
We sometimes denote $\det(A)$ by

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Geometrically, the cross product of two vectors in $\mathbb{R}^3$ can be represented as

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k.$$

Theorem 3.4

The area of a parallelogram with sides determined by the vectors $a = a_1 i + a_2 j$ and $b = b_1 i + b_2 j$ is

$$A = |\det(A)|,$$

where $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$.
The volume of a parallelepiped determined by the vectors $a = a_1 i + a_2 j + a_3 k$, $b = b_1 i + b_2 j + b_3 k$, $c = c_1 i + c_2 j + c_3 k$ is

$$V = |\det(A)|,$$

where $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$.

Proof. The area of the parallelogram is $A = bh = \|a\| h = \|a\| \|b\| \sin\theta = \|a \times b\|$. Since the k components of a and b are both zero, substitution yields

$$A = \|(a_1 b_2 - a_2 b_1) k\| = |a_1 b_2 - a_2 b_1| = |\det(A)|.$$

Similarly, the volume of the parallelepiped is $V = bh = \|b \times c\|h = \|b \times c\|\|a\||\cos\psi| = \|b \times c\||a \cdot n|$, where n is a unit vector that is perpendicular to the plane containing b and c. We now have

$$\begin{aligned} V &= \|b \times c\|\|a\||\cos\psi| = |a \cdot (b \times c)| \\ &= |(a_1 i + a_2 j + a_3 k) \cdot [(b_2 c_3 - b_3 c_2)i + (b_3 c_1 - b_1 c_3)j + (b_1 c_2 - b_2 c_1)k]| \\ &= |a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1)| \\ &= |\det(A)|. \end{aligned}$$

□

3.2 Properties of Determinants

For large values of n, evaluating a determinant of order n using the definition given in the previous section is not very practical. (Well, for me, evaluating a determinant of order 2 using that ugly definition is already not very practical :)) In the next sections, we develop alternative techniques (like real techniques) for evaluating determinants.

Theorem 3.5

If A is an $n \times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33}\dots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

Proof. Consider $\det(A) = \sum \sigma(p_1, p_2, \dots, p_n) a_{1_{p_1}} a_{2_{p_2}} \dots a_{n_{p_n}}$. If A is upper triangular, then $a_{ij} = 0$ whenever $i > j$, and the only nonzero terms in the preceding summation are those with $p_i \geqslant i$ for all i. Since all the p_i must be distinct, the only possibility is $p_i = i$, so the above equation reduces to

$$\det(A) = \sigma(1, 2, \dots, n) a_{11} a_{22} \dots a_{nn}.$$

Since $\sigma(1, 2, \dots, n) = 1$, it follows that

$$\det(A) = a_{11} a_{22} \dots a_{nn}.$$

The proof for the lower triangular matrices is analogous to the proof above. □

Now we turn our attention to the change of determinant from different operations: matrix algebra, elementary row operations, and more. We first let A be an $n \times n$ matrix, and discuss the elementary row operations case by case. Get the theoretical train honking.

(1) If B is the matrix obtained by permuting two rows of A, then

$$\det(B) = -\det(A).$$

Proof. Let B be the matrix obtained by interchanging row r and row s in A. Without the loss of generality, assume $r < s$. Then the elements of B are:

$$b_{ij} = a_{ij} \text{ if } i \neq r, s, \quad b_{ij} = a_{sj} \text{ if } i = r, \quad b_{ij} = a_{sj} \text{ if } i = s.$$

Thus, from the definition,

$$\begin{aligned}\det(B) &= \sum \sigma(p_1, ..., p_r, ..., p_s, ..., p_n) b_{1_{p_1}} ... b_{r_{p_r}} ... b_{s_{p_s}} ... b_{n_{p_n}} \\ &= \sum \sigma(p_1, ..., p_r, ..., p_s, ..., p_n) a_{1_{p_1}} ... a_{s_{p_r}} ... a_{p_{s_r}} ... a_{n_{p_n}} \\ &= -\sum \sigma(p_1, ..., p_s, ..., p_r, ..., p_n) a_{1_{p_1}} ... a_{r_{p_s}} ... a_{s_{p_r}} a_{n_{p_n}}\end{aligned}$$

Note that interchanging p_r and p_s in σ has the effect of changing the parity of the permutation. Note that the sum on the right-hand side of this equation is $\det(A)$, so that

$$\det(B) = -\det(A).$$

$\square$

(2) If B is the matrix obtained by multiplying one row of A by any scalar k, then

$$\det(B) = k\det(A).$$

Proof. Let B be the matrix obtained by multiplying the i-th row of A through by any scalar k. Then $b_{ij} = ka_{ij}$ for each j. Then

$$\begin{aligned}\det(B) &= \sum \sigma(p_1, ..., p_n) b_{1_{p_1}} ... b_{n_{p_n}} \\ &= \sum \sigma(p_1, ..., p_n) a_{1_{p_1}} ... (ka_{i_{p_i}}) ... a_{n_{p_n}} \\ &= k\det(A).\end{aligned}$$

$\square$

(3) If B is the matrix obtained by adding a multiple of any row of A to a different row of A, then

$$\det(B) = \det(A).$$

Proof. Note that (6) and (8) are necessary for the proof.
Let $A = \left[a_1, a_2, ..., a_n\right]^T$, and let B be the matrix obtained from A when k times row j of A is added to row i of A. Then

$$B = \left[a_1, a_2, ..., a_i + ka_j, ..., a_n\right]^T.$$

Using (6),

$$\det(B) = \det\left(\left[a_1, a_2, ..., a_i + ka_j, ..., a_n\right]^T\right) = \det\left(\left[a_1, a_2, ..., a_n\right]^T\right) + \det\left(\left[a_1, a_2, ..., ka_j, ..., a_n\right]^T\right).$$

Here, rows i and j of the second matrix are multiplies of one another, and so by (8), the value of the second of ther second determinant is zero. Thus,

$$\det(B) = \det\left(\left[a_1, a_2, ..., a_n\right]^T\right) = \det(A).$$

□

(4) For any scalar k and $n \times n$ matrix A, we have

$$\det(kA) = k^n \det(A).$$

Proof. Apply (2) to each row of the $n \times n$ matrix A. □

(5) $\det(A^T) = \det(A)$.

Proof. $\det(A^T) = \sum \sigma(p_1, \dots, p_n) a_{p_1 1} \dots a_{p_n n}$. Since $(p_1, \dots, p_n)$ is a permutation of $1, 2, \dots, n$, it follows that

$$a_{p_1 1} \dots a_{p_n n} = a_{1_{q_1}} \dots a_{n_{q_n}},$$

for appropriate values of $q_1, \dots, q_n$. Furthermore,

$$N(p_1, \dots, p_n) = N(q_1, \dots, q_n) \Rightarrow \sigma(p_1, \dots, p_n) = \sigma(q_1, \dots, q_n).$$

Substituting gives

$$\det(A^T) = \sum \sigma(q_1, \dots, q_n) a_{1_{q_1}} \dots a_{n_{q_n}} = \det(A).$$

□

(6) Let $a_1, a_2, \dots, a_n$ denote the row vectors of A. If the i-th row vector of A is the sum of two row vectors, say $a_i = b_i + c_i$, then $\det(A) = \det(B) + \det(C)$, where

$$B = \begin{bmatrix} a_1 \\ \vdots \\ a_{i-1} \\ b_i \\ a_{i+1} \\ \vdots \\ a_n \end{bmatrix} \quad C = \begin{bmatrix} a_1 \\ \vdots \\ a_{i-1} \\ c_i \\ a_{i+1} \\ \vdots \\ a_n \end{bmatrix}.$$

The corresponding property is also true for columns.

Proof. The elements of A are

$$a_{kj} = a_{kj} \text{ if } k \neq i, \quad a_{kj} = b_{kj} + c_{kj} \text{ if } k = i.$$

Thus,

$$\begin{aligned} \det(A) &= \sum \sigma(p_1, \dots, p_n) a_{1_{p_1}} \dots a_{n_{p_n}} \\ &= \sum \sigma(p_1, \dots, p_n) a_{1_{p_1}} \dots (b_{i_{p_i}} + c_{i_{p_i}}) \dots a_{n_{p_n}} \\ &= \sum \sigma(p_1, \dots, p_n) a_{1_{p_1}} \dots b_{i_{p_i}} \dots a_{n_{p_n}} + \sum \sigma(p_1, \dots, p_n) a_{1_{p_1}} \dots c_{i_{p_i}} \dots a_{n_{p_n}} \\ &= \det(B) + \det(C). \end{aligned}$$

□

(7) If A has a row (or column) of zeros, then $\det(A) = 0$.

Proof. Since each term in $\det(A)$ contains a factor from the row (or column) of zeros, each terms is zero. Hence, the $\det(A)$, the sum of the factors, is also 0. □

(8) If two rows (or columns) of A are scalar multiples of one another, then $\det(A) = 0$.

Proof. Assuming the rows and columns of A are all nonzero, and suppose rows i and j are scalar multiples of each other. More precisely, suppose that row j is j times row i for some $k \neq 0$. Let A' denote the matrix obtained by multiplying row i of the matrix A by k. By (2), $\det(A') = k\det(A)$. For A', row i and row j are identical. If we interchange these rows, the matrix is unaltered, but according to (1) the determinant of the resulting (unchanged) matrix is $-\det(A')$. Therefore,

$$\det(A') = -\det(A') \Rightarrow \det(A') = 0.$$

□

(9) $\det(AB) = \det(A)\det(B)$.

Proof. Let E denote an elementary matrix. Note that $\det(E) = -1$ if E permutes rows, $\det(E) = +1$ if E adds a multiple of one row to another row, and $\det(E) = k$ if E scales a row by K. Then, in each case, $\det(EA) = \det(E) = \det(A)$. Now consider two cases.

Case 1. If A is not invertible, then AB is also not invertible. Consequently, $\det(AB) = 0 = \det(A)\det(B)$. (This theorem will be covered later.)

Case 2. If A is invertible, then $A = E_1E_2...E_r$. It follows that

$$\begin{aligned}\det(AB) &= \det(E_1E_2...E_rB) = \det(E_1)\det(E_2...E_rB)\\ &= \det(E_1)\det(E_2)...\det(E_r)\det(B)\\ &= \det(E_1E_2...E_r)\det(B)\\ &= \det(A)\det(B).\end{aligned}$$

□

(10) If A is an invertible matrix, then $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof. Since A is invertible, $\det(A) \neq 0$. We can write $AA^{-1} = I_n$. Recalling that $\det(I_n) = 1$, using (9) gives

$$\det(A)\det(A^{-1}) = \det(I_n) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$

□

Note that for (9) and (10), we used a theorem that wasn't covered.

Theorem 3.6

Let A be an $n \times n$ matrix. Then, A is invertible if and only if $\det(A) \neq 0$.

Proof. Let A^* denote the reduced row-echelon form of A, and note that A is invertible if and only if $A^* = I_n$. Since A^* is obtained from A by a sequence of elementary row operations, (1)(2)(3) together imply that $\det(A)$ is a nonzero multiple of $\det(A^*)$. If A is invertible, then $\det(A^*) = \det(I_n) = 1$, so that $\det(A) \neq 0$.
Conversely, if $\det(A) \neq 0$, then $\det(A^*) \neq 0$. This implies that $A^* = I_n$, so A is invertible. □

As the linear system $Ax = b$ has unique solution for every $b \in \mathbb{R}^n$ if and only if A is invertible, the above theorem tells us that the system has a unique solution x if and only if $\det(A) \neq 0$. For the homogeneous $n \times n$ linear system $Ax = 0$, the system has an infinite number of solutions if and only if $\det(a) = 0$, and has only the trivial solution if and only if $\det(A) \neq 0$. More details will be covered in the invertible matrix theorem.

3.3 Cofactor Expansions

The underlying idea of cofactor expansion is that we can reduce a determinant of order n to a sum of determinants of order $n-1$. Repeating the process makes it possible to express any determinant as a sum of determinants of order 2. Before getting into the details of cofactor expansion, it is necessary to define minors and cofactors.

Definition 3.7: Minor

Let A be an $n \times n$ matrix. The **minor**, M_{ij}, of the element a_{ij} is the determinant of the matrix obtained by deleting the i-th row vector and j-th column vector of A.

Example 3.8. For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we have $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ and $M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$.

Definition 3.9: Cofactor

Let A be an $n \times n$ matrix. The **cofactor**, C_{ij}, of the element a_{ij} is defined by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the minor of a_{ij}.

From the above definition, we see that the cofactor of a_{ij} and the minor of a_{ij} are the same if $i + j$ is even, and they are opposite if $i + j$ is odd. The appropriate sign alternates as follows:

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

Example 3.10. For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we have $C_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ and $C_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$.

Theorem 3.11: Cofactor expansion theorem

Let A be an $n \times n$ matrix. If we multiply the elements in any row (or column) of A by their cofactors, then the sum of the resulting products is $\det(A)$. Thus, expanding along row i or column j gives

$$\det(A) = \sum_{k=1}^{n} a_{ij} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj}.$$

Proof. Consider $\det(A) = a_{i1}\hat{C}_{i1} + a_{i2}\hat{C}_{i2} + \ldots + a_{in}\hat{C}_{in}$, where the coefficients $\hat{C}_{ij}$ contain no elements from row i or column j. WE must show that

$$\hat{C}_{ij} = C_{ij},$$

where C_{ij} is the cofactor of a_{ij}.

Consider first a_{11}. From the original definition, the terms of $\det(A)$ that contain a_{11} are given by

$$a_{11} \sum \sigma(1, p_2, \ldots, p_n) a_{2_{p_2}} \ldots a_{n_{p_n}},$$

where the summation is over the $(n-1)!$ distinct permutations of $2, 3, \ldots, n$. Thus,

$$\hat{C}_{11} = \sum \sigma(1, p_2, \ldots, p_n) a_{2_{p_2}} \ldots a_{n_{p_n}}.$$

However, this summation is just the minor M_{11}. Since $C_{11} = M_{11}$, we have shown that the coefficient of a_{11} in $\det(A)$ is indeed the cofactor C_{11}.

Now consider the element a_{ij}. By successively interchanging adjacent rows and columns of A, we can move a_{ij} into the (1,1) position without altering the relative positions of the other rows and columns of A. Denoting the resulting matrix as A', obtaining A' from A requires $i-1$ row interchanges and $j-1$ column interchanges, so the total number of interchanges required to obtain A' from A is $i+j-2$. Consequently,

$$\det(A) = (-1)^{i+j-2}\det(A') = (-1)^{i+j}\det(A').$$

The coefficient of a_{ij} in $\det(A)$ must be $(-1)^{i+j}$ times the coefficient of a_{ij} in $\det(A')$. As a_{ij} occurs in the (1,1) position of A', its coefficient in $\det(A')$ is M'_{11}. Since the relative positions of the remaining rows in A has not altered, it follows that $M'_{11} = M_{ij}$, and therefore the coefficient of a_{ij} in $\det(A')$ is M_{ij}. Consequently, the coefficient of a_{ij} in $\det(A)$ is $(-1)^{i+j} M_{ij} = C_{ij}$. Applying this result to each elements gives

$$\hat{C}_{ij} = C_{ij},$$

which established the theorem for expansion along a row of elements $a_{i1}, a_{i2}, \ldots, a_{in}$. The result along a column follows directly as $\det(A^T) = \det(A)$. □

Example 3.12. We want to evaluate $\begin{vmatrix} 2 & 1 & 8 & 6 \\ 1 & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 2 \end{vmatrix}$. To do so, we

$$\begin{vmatrix} 2 & 1 & 8 & 6 \\ 1 & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -7 & 6 & 0 \\ 1 & 4 & 1 & 3 \\ 0 & 6 & 2 & 7 \\ 0 & -1 & -2 & -1 \end{vmatrix} = -\begin{vmatrix} -7 & 6 & 0 \\ 6 & 2 & 7 \\ -1 & -2 & -1 \end{vmatrix} = -\begin{bmatrix} -7 & 6 & 0 \\ -1 & -12 & 0 \\ -1 & -2 & -1 \end{bmatrix} = \begin{vmatrix} -7 & 6 \\ -1 & -12 \end{vmatrix} = 90.$$

Some short commentary: the first step reduces the first column, then we used cofactor expansion along column 1. Then for the order 3 determinant, reducing the third column and using cofactor expansion along column 3 gives an order 2 determinant, which can be calculated with ease.

Now we introduce the adjoint method for finding the inverse of A, as a corollary to the cofactor expansion theorem.

Corollary 3.13

If the elements in the i-th row (or column) of an $n \times n$ matrix A are multiplied by the cofactors of a different row (or column), then the sum of the resulting products is zero. That is,

$$\sum_{k=1}^{n} a_{ik}C_{jk} = 0 \qquad \sum_{k=1}^{n} a_{ki}C_{kj} = 0 \quad i \neq j.$$

Proof. We prove the first equation. Let B be the matrix obtained from A by adding row i to row j in matrix A. We know that $\det(B) = \det(A)$, and cofactor expansion of B along row j gives

$$\det(A) = \det(B) = \sum_{k=1}^{n}(a_{jk} + a_{ik})C_{jk} = \sum_{k=1}^{n} a_{jk}C_{jk} + \sum_{k=1}^{n} a_{ik}C_{jk}$$

This gives

$$\det(A) = \det(A) + \sum_{k=1}^{n} a_{ik}C_{jk},$$

and the corollary follows immediately. The second equation can be proven by similar manner. □

Definition 3.14: Adjoint

If every element in an $n \times n$ matrix is replaced by its cofactor, the resulting matrix is the **matrix of cofactors**, denoted as M_C. The transpose of the matrix of cofactors, M_C^T, is called the **adjoint** of A and is denoted $\operatorname{adj}(A)$. The elements of $\operatorname{adj}(A)$ are

$$\operatorname{adj}(A)_{ij} = C_{ji}.$$

Example 3.15. Consider $A = \begin{bmatrix} 6 & -1 & 0 \\ 2 & -2 & 1 \\ 3 & 0 & -3 \end{bmatrix}$. The cofactors of A are

$$C_{11} = 6, C_{12} = 9, C_{13} = 6, C_{21} = -3, C_{22} = -18, C_{23} = -3, C_{31} = -1, C_{32} = -6, C_{33} = -10.$$

Thus, we have

$$M_C = \begin{bmatrix} 6 & 9 & 6 \\ -3 & -18 & -3 \\ -1 & -6 & -10 \end{bmatrix} \Rightarrow \text{adj}(A) = \begin{bmatrix} 6 & -3 & -1 \\ 9 & -19 & -6 \\ 6 & -3 & -10 \end{bmatrix}.$$

Theorem 3.16: The adjoint method

If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A).$$

Proof. Let $B = \frac{1}{\det(A)}\text{adj}(A)$. It suffices to show that $AB = I_n = BA$. Using the index form of the matrix product, we have

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} a_{ij} \cdot \frac{1}{\det(A)} \cdot \text{adj}(A)_{kj} = \frac{1}{\det(A)} \sum_{k=1}^{n} a_{ik}C_{jk} = \delta_{ij}.$$

Here, the last step uses the corollary

$$\sum_{k=1}^{n} a_{ik}C_{jk} = \delta_{ij}\det(A) \qquad \sum_{k=1}^{n} a_{ki}C_{kj} = \delta_{ij}\det(A).$$

The statement that $BA = I_n$ can be proven analogously. □

At last, we introduce Cramer's rule, which is a very useful tool to solve $n \times n$ systems of equations.

Theorem 3.17: Cramer's rule

It $\det(A) \neq 0$, the unique solution to the $n \times n$ system $Ax = b$ is $(x_1, x_2, \ldots, x_n)$, where

$$x_k = \frac{\det(B_k)}{\det(A)}, \qquad k = 1, 2, \ldots, n,$$

where

$$B_k = \begin{bmatrix} a_{11} & a_{12} & \ldots & b_1 & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & b_2 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & b_n & \ldots & a_{nn} \end{bmatrix}.$$

Proof. If $\det(A) \neq 0$, then the system $Ax = b$ has the unique solution

$$x = A^{-1}b,$$

where, from the adjoint method, we have

$$x = \frac{1}{\det(A)}\operatorname{adj}(A)b.$$

Here, letting

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

and using $\operatorname{adj}(A)_{ij} = C_{ji}$ gives

$$x_k = \sum_{i=1}^{n}(A^{-1})_{ki}b_i = \sum_{i=1}^{n}\frac{1}{\det(A)}\operatorname{adj}(A)_{ki}b_i = \frac{1}{\det(A)}\sum_{i=1}^{n}C_{ik}b_i, \quad k = 1, 2, \dots, n.$$

The right-hand sum is exactly the cofactor expansion of $\det(B_k)$, so we arrive at the conclusion that

$$x_k = \frac{\det(B_k)}{\det(A)}, \quad k = 1, 2, \dots, n.$$

□

Again, Cramer's rule requires more work than the Gaussian Elimination method and only works for $n \times n$ systems whose coefficient matrix is invertible. I will not present an applicational example here.

4 Vector Spaces

Suppose we wish to find solutions to the differential equation $y' = 2y = 0$. The results can be expressed in the form $y(x) = ce^{-2x}$ for some constant c. Similarly, (in chapter 8) we will know that every solution to the homogeneous second-order differential equation $y'' + a_1 y' + a_2 y = 0$ has the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$.
The theory underlying the solution to a linear differential equation and the theory underlying the solution of linear equations can be considered as special cases of solving linear problems.
We begin developing the way of formulating linear problems in terms of abstract set of vectors V.

4.1 Vectors in $\mathbb{R}^n$

A geometric vector can be considered as a directed line segment with a magnitude (length) and a direction.
Vectors are nice, that they follow certain properties. Namely, the commutative property $x + y = y + x$ and the associative property $x + (y + z) = (x + y) + z$. Additively, the existence of the zero vector $x + 0 = x$ and the additive inverse $x + (-x) = 0$ together forms the fundamental properties of vector addition.
For scalar multiplications of vectors, define kx as the vector with magnitude $|k|x$ and direction dependent on k. Vectors also follow multiplicative properties, such as the associative property $(st)x = s(tx)$, the distributive property $r(x + y) = rx + ry$ and $(s + t)x = sx + tx$, as well the existence of the one scalar $1x = x$.
Now consider the components of the geometric vector $v \in \mathbb{R}^n$. As a natural extension of addition and scalar multiplication, the following properties hold true in $\mathbb{R}^n$ for all $v = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$:

$$v + w = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$kv = (kx_1, kx_2, ..., kx_n)$$

4.2 Definition of a Vector Space

Definition 4.1: Vector space

Let V be a nonempty set whose elements are called vectors. Consider an addition operation and a scalar multiplication operation. V is a **vector space** over f if the following conditions are satisfied:

(a) Closure under addition. For $u, v \in V$, $u + v \in V$.

(b) Closure under scalar multiplication. For $u \in V$ and $k \in \mathbb{R}$, $ku \in V$.

There are actually more conditions to the vector spaces, but often times showing closure will suffice.

Example 4.2: Examples of vector spaces.

(1) $\mathbb{R}^n$ and $\mathbb{C}^n$, the real/complex vector space of real/complex numbers.

(2) $M_{m\times n}(\mathbb{R})$, the real vector space of all $m \times n$ matrices.

(3) $C^k(I)$, the vector space of all real-valued functions that are continuous and is k-times differentiable on an interval.

(4) $P_n(\mathbb{R})$, the vector space of all real-valued polynomials of maximum degree of n with real coefficients, i.e., $P_n(\mathbb{R}) = \left\{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\right\}$.

4.3 Subspaces

Consider a subset of vectors from an appropriate vector spaces. This raises a key question, that whether this subset of vectors is a vector space in its own right.

Definition 4.3: Subspace

Let S be a nonempty subset of a vector space V. If S is itself a vector space under the same operations of addition and scalar multiplication as used in V, then S is a **subspace** of V.

Theorem 4.4

Let S be a nonempty subset of a vector space V. S is a subspace of V if and only if S is closed under the operations of addition and scalar multiplication in V.

Proof. We prove the forward direction first. If S is a subspace of V, then it is a vector space, so it is closed under addition and scalar multiplication by definition. Conversely, assume that S is closed under addition and multiplication. Since we use the same operations in S as in V, the vector space axioms are inherited from V by the subset S, so S is a subspace of V. □

Example 4.5. Let $V = \mathbb{R}^2$, and let $S_1 = \{(x, x-1) : x \in \mathbb{R}\}$, $S_2 = \left\{(x, x^2) : x \in \mathbb{R}\right\}$. We want to show if S_1 and S_2 are subspaces of V. We easily know that S_1 is not a subspace of V by the zero vector check. Although S_2 satisfies the zero vector check, it is not closed under addition. Consider $x = (x_1, x_2), y = (y_1, y_2)$, and $x, y \in S_2$. $x + y = (x_1 + y_1, x_2^2 + y_2^2) \notin S_2$.

Theorem 4.6

$S = \{0\}$ is a subspace V if $S \subset V$.

Proof. S is nonempty, and it is trivial that S is closed under addition and scalar multiplication. □

Definition 4.7: Null space

Let A be an $m \times n$ matrix. The solution set to the corresponding homogeneous linear system $Ax = 0$ is the **null space** of A, and is denoted nullspace(A), i.e.,

$$\text{nullspace}(A) = \{x : Ax = 0\}.$$

We now show the connection between differential equations and vector spaces.

Theorem 4.8

The set of all solutions to the homogeneous linear differential equation

$$y'' + a_1(x)y' + a_2(x)y = 0$$

on an interval I is a vector space.

Proof. Let S denote the set of all solutions to the given differential equation, a nonempty subset of $C^2(I)$. Assume $y_1, y_2 \in S$, $k \in \mathbb{R}$. Then,

$$y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0 \quad y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0$$

Now, if $y(x) = y_1(x) + y_2(x)$, then $y'' + a_1y' + a_2y = (y_1 + y_2)'' + a_1(y_1 + y_2)' + a_2(y_1 + y_2) = 0$.
Also, if $y(x) = ky_1(x)$, then $y'' + a_1y' + a_2y = (ky_1)'' + a_1(ky_1)' + a_2(ky_1) = 0$.
As S is closed under addition and scalar multiplication, S is a subspace of $C^2(I)$. □

4.4 Spanning Sets

Before introducing spanning sets, we explain the concept of a linear combination of $v_1, v_2, \ldots, v_k$, the most general way in which we can combine the vectors $v_1, v_2, \ldots, v_k \in V$:

$$v = c_1v_1 + c_2v_2 + \ldots + c_kv_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Definition 4.9: Spanning sets

If every vector in a vector space V can be written as a linear combination of $v_1, v_2, \ldots, v_k$, then V is spanned or generated by $v_1, v_2, \ldots, v_k$ and call the set of vectors $\{v_1, v_2, \ldots, v_k\}$ a **spanning set** of V.

Theorem 4.10

Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Then $\{v_1, v_2, \ldots, v_k\}$ spans $\mathbb{R}^n$ if and only if the matrix $A = [v_1, v_2, \ldots, v_k]$, the linear system $Ac = v$ is consistent for every $v \in \mathbb{R}^n$.

Proof. Rewriting the system as the linear combination $c_1v_1 + c_2v_2 + \ldots + c_kv_k = v$. The existence of a solution $(c_1, c_2, \ldots, c_k)$ to this vector equation for each $v \in \mathbb{R}^n$ is equivalent to $\{v_1, v_2, \ldots, v_k\}$ spans $\mathbb{R}^n$. □

Example 4.11. We want to determine a spanning set for $P_2(\mathbb{R})$.
Consider $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$. Then, $p(x) = a_0p_0(x) + a_1p_1(x) + a_2p_2(x)$.

Theorem 4.12

Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V. Then span $\{v_1, v_2, \ldots, v_k\}$ is a subspace of V.

Proof. Consider $v, w \in S$. Then we have

$$v = a_1v_1 + a_2v_2 + ... + a_kv_k \quad w = b_1v_1 + b_2v_2 + ... + b_kv_k.$$

Thus, $v + w = (a_1v_1 + a_2v_2 + ... + a_kv_k) + (b_1v_1 + b_2v_2 + ... + b_kv_k) = c_1v_1 + c_2v_2 + ... + c_kv_k$.
Also, $kv = ka_1v_1 + ka_2v_2 + ... + ka_kv_k = d_1v_1 + d_2v_2 + ... + d_kv_k$.
As S is closed under addition and scalar multiplication, S is a subspace of V. □

4.5 Linear Dependence and Linear Independence

Definition 4.13: Linear dependency

A finite nonempty set of vectors $\{v_1, v_2, ..., v_k\}$ in a vector space V is said to be linearly dependent if there exist scalars $c_1, c_2, ..., c_k$, not all zero, such that

$$c_1v_1 + c_2v_2 + ... + c_kv_k = 0.$$

Such a nontrivial linear combination of vectors is sometimes referred to as a linear dependency among the vectors $v_1, v_2, ..., v_k$.
A set of vectors that is not linearly dependent is called linearly independent.

Example 4.14. Let V be the vector space of all functions defined on an interval I. If

$$f_1(x) = 1 \quad f_2(x) = 2\sin^2 x \quad f_3(x) = -5\cos^2(x),$$

then $\{f_1, f_2, f_3\}$ is linearly dependent in V, since the trigonometric identity implies $f_1(x) = f_2(x)/2 - f_3(x)/5$. We can therefore conclude from theorem 4.5.2 that

$$\text{span}\left\{1, 2\sin^2 x, -5\cos^2 x\right\} = \text{span}\left\{2\sin^2 x, -5\cos^2 x\right\}$$

Now we consider linear dependency in $\mathbb{R}^n$. Let $\{v_1, v_2, ..., v_k\}$ be a set of vectors in $\mathbb{R}^n$. Let A denote the matrix $A = [v_1, v_2, ..., v_k]$. Since each of the given vectors is in $\mathbb{R}^n$, it follows that A is a $n \times k$ matrix. The linear combination $c_1v_1 + c_2v_2 + ... + c_kv_k = 0$ can be written as $Ac = 0$, where $c = [c_1c_2...c_k]^T$.
Then, let $v_1, v_2, ..., v_k$ be vectors in $\mathbb{R}^n$ and $A = [v_1, v_2, ..., v_k]$. Then $\{v_1, v_2, ..., v_k\}$ is linearly dependent if and only if the linear system $Ac = 0$ has a nontrivial solution for c. That is, $\det(A) = 0$.

We now consider the linear dependency of the set of functions.

Definition 4.15: Linear dependency of functions

The set of functions $\{f_1, f_2, ..., f_k\}$ is linearly independent on an interval I if and only if the only values of the scalars $c_1, c_2, ..., c_k$ such that

$$c_1f_1(x) + c_2f_2(x) + ... + c_kf_k(x) = 0$$

are $c_1 = c_2 = ... = c_k = 0$ for all $x \in I$.

The linear dependency of functions can be determined by the Wronskian.

Definition 4.16: Wronskian

Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. The **Wronskian** of these functions is the order k determinant defined by

$$W[f_1, f_2, \ldots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_k(x) \\ f_1'(x) & f_2'(x) & \ldots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \ldots & f_k^{(k-1)}(x) \end{vmatrix}.$$

Theorem 4.17

Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. If $W[f_1, f_2, \ldots, f_k]$ is nonzero at some point $x_0 \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is linearly independent on I.

Proof. Assume $c_1 f_1(x) + c_2 f_2(x) + \ldots + c_k f_k(x) = 0$ for all $x \in I$. Differentiating $k-1$ times yields the linear system whose the determinant of the matrix of coefficients is $W[f_1, f_2, \ldots, f_k](x)$. Consequently, if $W[f_1, f_2, \ldots, f_k](x_0) \neq 0$ for some $x_0 \in I$, then the determinant is nonzero, and therefore the only solution is the trivial solution $c_1 = c_2 = \ldots = c_k = 0$. That is, the given set is linearly independent on I. □

The Wronskian can only be used to determine if a set of functions is linear independent. That is, if $W[f_1, f_2, \ldots, f_k](x) = 0$ for all $x \in I$, we cannot conclude any information as to the linear dependence or independence of $\{f_1, f_2, \ldots, f_k\}$ on I.

4.6 Bases and Dimension

Definition 4.18: Basis

A set of vectors $\{v_1, v_2, \ldots, v_k\} \in V$ is called a **basis** for V if

(a) the vectors $v_1, v_2, \ldots, v_k$ are linearly independent.

(b) the vectors together span V.

There do exist vector spaces V for which it is impossible to find a finite set of linearly independent vectors that span V. For example, the vector space $C^n(I)$ have infinitely many linearly independent vectors that span V. These are called **infinite-dimensional vector spaces**. We primarily consider the vector spaces that can be spanned by finitely many vectors, or finite-dimensional vector spaces.

Theorem 4.19

If a finite-dimensional vector space has a basis of n vectors, then any set of more than n vectors is linearly dependent.

Proof. Let $v_1, v_2, ..., v_n$ be a basis for V, and any set of $m > n$ vectors. Consider $\{u_1, u_2, ..., u_m\}$, each can be represented as a linear combination of $v_1, v_2, ..., v_n$. The system of equation $c_1 u_1 + c_2 u_2 + ... + c_m u_m = 0$ has nontrivial solutions as the number of equations exceeds the number of unknowns. Hence, the vectors $u_1, u_2, ..., u_m$ is necessarily linearly dependent. □

Definition 4.20: Dimension

The dimension of a finite-dimensional vector space V, written $\dim[V]$, is the number of vectors in any basis for V, except for when $V = \{0\}$, where its dimension is zero.

Example 4.21. The dimensions for the following common vector spaces are trivial.

(a) $\dim[\mathbb{R}^n] = n$

(b) $\dim[M_{m\times n}(\mathbb{R})] = mn$

(c) $\dim[P_n(\mathbb{R})] = n + 1$

(d) $\dim[C^k(I)] = \infty$

Theorem 4.22

If $\dim[V] = n$, then any set of n linearly independent vectors in V is a basis for V.

Proof. Let $v_1, v_2, ..., v_n$ be n linearly independent vectors in V. We need to show that they span V.
Consider any $v \in V$, then the equation $c_0 v + c_1 v_1 + ... + c_n v_n = 0$ has at least one non-trivial solution as $\{v, v_1, v_2, ..., v_n\}$ is linearly dependent. Hence, $v = -(c_1 v_1 + v_2 v_2 + ... + c_n v_n)/c_0$. Likewise, every vector v can be written as a linear combination of $v_1, v_2, ..., v_n$ and hence $\{v_1, v_2, ..., v_n\}$ spans V. □

The result of this proof is significant, especially when connecting with differential equations. In later chapters, we will explicitly construct a basis for the solution space to the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y = 0.$$

We can now establish an equivalence relation between statements. If $\dim[V] = n$ and $S = \{v_1, v_2, ..., v_n\}$ is a set of n vectors in V, then the following statements are equivalent:

(1) S is a basis for V.

(2) S is linearly independent.

(3) S spans V.

4.7 Change of Basis

If we have a finite basis for a vector space V, then, since the vectors in a basis span V, any vector in V can be expressed as a linear combination of the basis vectors. The next theorem establishes that there is only one way in which we can do this.

Theorem 4.23

If V is a vector space with basis $\{v_1, v_2, \ldots, v_n\}$, then every vector $v \in V$ can be written uniquely as a linear combination of $v_1, v_2, \ldots, v_n$.

Proof. Since $v_1, v_2, \ldots, v_n$ span V, every vector $v \in V$ can be expressed as

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n.$$

Suppose also that

$$v = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n,$$

it suffices to prove that $a_i = b_i$ for each i. Consider

$$v - v = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \ldots + (a_n - b_n)v_n = 0.$$

However, as $v_1, v_2, \ldots, v_n$ are linearly independent, the only solution to the equation is the trivial solution $a_i - b_i = 0$. This implies $a_i = b_i$ for each i, completing the proof. $\square$

Definition 4.24: Ordered basis

An **ordered basis** is defined as a basis in which the order is kept track of.
If $B = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis for V and $v \in V$, then the scalars $c_1, c_2, \ldots, c_n$ in the unique n-tuple such that

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

are called the components of v relative to the ordered basis $B = \{v_1, v_2, \ldots, v_n\}$. The column vector of the components of v relative to the ordered basis by $[v]_B$, and $[v]_B$ is the component vector of v relative to B.

Example 4.25. We want to determine $[v]_B$ of $v = (1, 7)$ in $\mathbb{R}^3$ relative to $B = \{(1, 2), (3, 1)\}$.
Letting $v_1 = (1, 2)$ and $v_2 = (3, 1)$ allows us to determine constants c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

The solution is $(4, -1)$, which gives the components of v relative to the ordered basis $B = \{v_1, v_2\}$. Thus, $v = 4v_1 - v_2$, giving $[v]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

If we are given two different ordered basis for an n-dimensional vector space V, say

$$B = \{v_1, v_2, \ldots, v_n\} \quad C = \{w_1, w_2, \ldots, w_n\},$$

and a vector $v \in V$, we want to know the relation between $[v]_B$ and $[v]_C$. The connection is defined as a **change-of-basis matrix**, as follows.

Definition 4.26: Change of basis matrix

Let V be an n-dimensional vector space with ordered basis $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_n\}$. The **change-of-basis matrix** is defined by

$$P_{C \leftarrow B} = [[v_1]_C, [v_2]_C, \ldots, [v_n]_C].$$

In words, the components of each vector in the old basis B with respect of the new basis C and write the component vectors in the columns of the change of basis matrix.
There then exists a relation between the component vector:

$$[v]_C = P_{C \leftarrow B}[v]_B.$$

Example 4.27. Let $V = \mathbb{R}^2$, $B = \{(1,2),(3,4)\}$, $C = \{(7,3),(4,2)\}$, $v = (1,0)$. We want to know: $[v]_B$ and $[v]_C$, $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$, and $[v]_C$ from the above equation.
Consider $c_1(1,2) + c_2(3,4) = (1,0)$, and $d_1(7,3) + d_2(4,2) = (1,0)$. Then, we have $(c_1, c_2) = (-2,1)$, $(d_1, d_2) = (1, -3/2)$ as the solution sets. Then,

$$[v]_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad [v]_C = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}.$$

Consider $P_{C \leftarrow B}$, we have

$$[v_1]_C = \begin{bmatrix} -3 \\ 5.5 \end{bmatrix} \quad [v_2]_C = \begin{bmatrix} -5 \\ 9.5 \end{bmatrix} \quad P_{C \leftarrow B} = \begin{bmatrix} -3 & -5 \\ 5.5 & 9.5 \end{bmatrix}.$$

Consider $P_{B \leftarrow C}$, we have

$$[w_1]_B = \begin{bmatrix} -9.5 \\ 5.5 \end{bmatrix} \quad [w_2]_B = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \quad P_{B \leftarrow C} = \begin{bmatrix} -9.5 & -5 \\ 5.5 & 3 \end{bmatrix}.$$

Then, we compute as follows:

$$P_{C \leftarrow B}[v]_B = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} = [v]_C.$$

Theorem 4.28

Let V be a vector space with ordered bases A, B, C. Then

$$P_{C \leftarrow A} = P_{C \leftarrow B} P_{B \leftarrow A}.$$

Proof. For every $v \in V$, we have

$$P_{C \leftarrow B} P_{B \leftarrow A}[v]_A = P_{C \leftarrow B}[v]_B = [v]_C = P_{C \leftarrow A}[v]_A.$$

□

4.8 Row Space and Column Space

Definition 4.29: Row space and Column space

Let $A = [a_{ij}]$ be an $m \times n$ real matrix. The row vectors of this matrix are row n-vectors, so they together span a subspace of $\mathbb{R}^n$. The **row space** of A is the subspace of $\mathbb{R}^n$.
Similarly, the column vectors of this matrix are row m- vectors, so they together span a subspace of $\mathbb{R}^m$. The **column space** of A is the subspace of $\mathbb{R}^m$.

Theorem 4.30

Let A be an $m \times n$ matrix. The set of column vectors of A corresponding to those column vectors containing leading ones in any row-echelon form of A is a basis for colspace(A).

Example 4.31. Determine a basis for colspace(A) if

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 2 & 4 & -2 & -3 & -1 \\ 5 & 10 & -5 & -3 & -1 \\ -3 & -6 & 3 & 2 & 1 \end{bmatrix}.$$

Reducing A to row-echelon form gives

$$A \sim \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The column vectors are then the first, fourth, and fifth rows, so a basis for colspace(A) is

$$\{(1, 2, 5, -3), (-2, -3, -3, 2), (-1, -1, -1, 1)\}.$$

4.9 The Rank-Nullity Theorem

Definition 4.32: Nullity

The **nullity** of A is the dimension of the null space of A.

Theorem 4.33: Rank-Nullity Theorem

For any $m \times n$ matrix A,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Proof. We don't present a direct proof here. Refer to the general rank-nullity theorem in chapter 6. □

4.10 Invertible Matrix Theorem II

We now present perhaps the most important theorem of this book: the full Invertible Matrix Theorem. This time, the full one. If you are only allowed to bring one sheet of paper to your linear algebra exam, this is undoubtedly the page (unless your exam is differential-equation-heavy, then I guess not :))

Theorem 4.34: Invertible matrix theorem

Let A be an $n \times n$ matrix. The following conditions on A are equivalent:

(a) A is invertible.

(b) The equation $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

(c) The equation $Ax = 0$ has only the trivial solution $x = 0$.

(d) $\text{rank}(A) = n$.

(e) A can be expressed as a product of elementary matrices.

(f) A is row-equivalent to I_n.

(g) $\text{nullity}(A) = 0$.

(h) $\text{nullspace}(A) = \{0\}$.

(i) The columns of A form a linearly independent set of vectors in $\mathbb{R}^n$.

(j) $\text{colspace}(A) = \mathbb{R}^n$.

(k) The columns of A form a basis for $\mathbb{R}^n$.

(l) The rows of A form a linearly independent set of vectors in $\mathbb{R}^n$.

(m) $\text{rowspace}(A) = \mathbb{R}^n$.

(n) The rows of A form a basis for $\mathbb{R}^n$.

(o) A^T is invertible.

Proof. The equivalence of (a)(b)(c)(d)(e)(f) has been established in section 2.8.
The equivalence of (a)(h) has been proved in the Rank-Nullity theorem. The equivalence of (g)(h) is trivial. The equivalence of (a)(i) is immediate from section 4.5. Since the dimension of $\text{colspace}(A)$ is simply $\text{rank}(A)$, the equivalence of (a)(j) is immediate. Next, from the definition of a basis, (i)(j)(k) are logically equivalent. Moreover, since the row space and column space of A always have the same dimension, (j)(m) are equivalent. Since $\text{rowspace}(A) = \text{colspace}(A^T)$, the equivalence of (j)(m) implies the equivalence of (a)(o). Finally, the equivalence of (a)(o) proves that (k)(n) are equivalent, and (i)(l) are equivalent.

Yes, at last, everything is equivalent. How magical it is. □

6 Linear Transformation

A variety of problems we have studied to this point in the text, both in linear algebra and in differential equations, can be viewed as special cases of the general problem of finding all vectors v in a vector space with the property that $T(v) = 0$, where T is a mapping from a vector space V into a vector space W.

6.1 Definition of a Linear Transformation

Definition 6.1: Mapping

Let V and W be vector spaces. A **mapping** from V into W is a rule that assigns to each vector $v \in V$ precisely one vector $w = T(v) \in W$. Such mapping is denoted as $T : V \to W$.

Definition 6.2: Linear transformation

Let V and W be vector spaces. A mapping $T : V \to W$ is a **linear transformation** from V to W if:

(1) $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.

(2) $T(cv) = cT(v)$ for all $v \in V$ and all $c \in \mathbb{R}$.

These properties are the linearity properties. V is the domain of T, and W is the codomain of T.

Now we give an example of a linear transformations.

Example 6.3. Define $T : C^2(I) \to C^0(I)$ by $T(y) = y'' + y$. Verify that T is a linear transformation.
Consider $y_1, y_2 \in C^2(I)$. Then $T(y_1 + y_2) = (y_1 + y_2)'' + (y_1 + y_2) = y_1'' + y_1 + y_2'' + y_2 = T(y_1) + T(y_2)$.
Now consider $y_1 \in C^2(I)$. Then $T(cy_1) = (cy_1)'' + cy_1 = c(y_1'' + y_1) = cT(y_1)$.
Both linearity properties are satisfied, so T is a linear transformation.

Theorem 6.4: Linear transformation and combination

A mapping $T : V \to W$ is a linear transformation if and only if

$$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$$

for all $v_1, v_2 \in V$ and all scalars c_1, c_2.

Proof. If $T(c_1v_1+c_2v_2) = c_1T(v_1)+c_2T(v_2)$, then the linearity properties, which are special cases of the equation, are satisfied. ((1) by $c_1 = c_2 = 1$, (2) by $c_1 = c$ and $c_2 = 0$) Hence, T is a linear transformation.
Conversely, if T is a linear transformation, then $T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2) = c_1T(v_1) + c_2T(v_2)$. □

Now we are interested in linear transformations between vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, as they are very pivotal in linear algebra and its applications.

Theorem 6.5: $T(x) = Ax$ as a linear transformation

Let A be an $m \times n$ matrix, and define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(x) = Ax$. Then T is a linear transformation.

Proof. We want to verify the linearity properties by considering $x, y \in \mathbb{R}^n$.
$T(x+y) = A(x+y) = Ax + Ay = T(x) + T(y)$, and $T(cx) = A(cx) = cA(x) = cT(x)$. As both linearity properties are satisfied, then T is a linear transformation. In fact, T is often called a **matrix transformation**. □

Example 6.6. We want to determine the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ if

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ -5 & 3 \\ 0 & -4 \end{bmatrix}.$$

To determine the transformation, we have

$$T(x) = Ax = \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ -5 & 3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 3x_1 - x_2 \\ -5x_1 + 3x_2 \\ -4x_2 \end{bmatrix}$$

Here, we have $T(x_1, x_2) = (2x_1 + x_2, 3x_1 - x_2, -5x_1 + 3x_2, -4x_2)$.

After we have worked the direct way, we also want to understand how the converse works. That is, given a certain transformation T, we want to find out the matrix of transformation.

Definition 6.7: Matrix of transformation

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the $m \times n$ matrix

$$A = [T(e_1), T(e_2), \dots, T(e_n)]$$

is called the **matrix** of T.

Example 6.8. We want to determine the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T(x_1, x_2, x_3) = (-1 + 3x_3, -2x_3, 2x_1 + 5x_2 - 9x_3, -7x_1 + 5x_2).$$

We consider standard basis vectors in $\mathbb{R}^3$: $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. This gives $T(e_1) = (-1, 0, 2, -7)$, $T(e_2) = (0, 0, 5, 5)$, $T(e_3) = (3, -2, -9, 0)$. The matrix of the transformation is

$$A = [T(e_1), T(e_2), T(e_3)] = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & -2 \\ 2 & 5 & -9 \\ -7 & 5 & 0 \end{bmatrix}.$$

6.2 Transformations of $\mathbb{R}^2$

We consider the particular case of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ in this section. Often called a **transformation of** $\mathbb{R}^2$, this transformation can be represented by its effect on an arbitrary point in the Cartesian plane.
Consider a line

$$l : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} a \\ b \end{pmatrix} = x_1 + tv.$$

The transformation $T(x) = Ax$ therefore transforms the line into

$$T(x) = A(x_1 + tv) = Ax_1 + tAv = y_1 + tw.$$

Here, $y_1 = Ax_1$ and $w = Av$.
We now show some simple transformations in $\mathbb{R}^2$, considering $v = (x, y)$ is an arbitrary point in $\mathbb{R}^2$.

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_{xy} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The above three transformations reflect a point $v = (x, y)$ over the x-axis, y-axis, and the line $y = x$, respectively.

$$LS_x = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad LS_y = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

The above two stretch a point $v = (x, y)$. The first gives $(x, y) \to (kx, y)$, and the second $(x, y) \to (x, ky)$. If we only consider the situations where $k > 0$, then the transformation can be either an expansion (when $k > 1$), a compression (when $k < 1$), or a identity transformation (when $k = 1$).

$$S_x = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad S_y = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

The above two shear a point $v = (x, y)$. The first gives $(x, y) \to (x + ky, y)$, and the second $(x, y) \to (x, kx + y)$. In this case, each point in the plane is moved parallel to the axis a distance proportional to its other coordinate.
Consider T as any transformation of $\mathbb{R}^2$ with invertible matrix A. Then,

$$T(v) = Av = E_1^{-1} E_2^{-1} \ldots E_n^{-1} v.$$

This means, if we consider any transformation with an invertible matrix, we can describe the transformation T as a combination of reflections, shears, and stretches.

Example 6.9. Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ with invertible matrix $A = \begin{bmatrix} 3 & 9 \\ 1 & 2 \end{bmatrix}$. We want to describe T as a combination of reflections, shears, and stretches.
We do this by first reducing A to reduced row-echelon form.

$$\begin{bmatrix} 3 & 9 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding elementary matrices are then

$$P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_{12}(-3) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad M_2(1/3) = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \quad A_{21}(-2) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

We can now write T as

$$T(v) = Av = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} v.$$

Here, we know that T consists of a shear parallel to x-axis, followed by a stretch in the y-direction, followed by a shear parallel to the y-axis, rounded up by a reflection in $y = x$.

6.3 The Kernel and Range of a Linear Transformation

If $T : V \to W$ is any linear transformation, there is an associated homogeneous linear vector equation $T(v) = 0$.

Definition 6.10: Kernel

Let $T : V \to W$ be a linear transformation. The set of all vectors $v \in V$ such that $T(v) = 0$ is called the **kernel** of T. Mathematically,

$$\text{Ker}(T) = \{v \in V : T(v) = 0\}.$$

We see that the concept of kernel is exactly same as null space. We often use kernel in linear transformations and null space in vector spaces, but they can be interchangeable.

Definition 6.11: Range

The **range** of the linear transformation $T : V \to W$ is the subset of W consisting of all transformed vectors from V. Mathematically,

$$\text{Rng}(T) = \{T(v) : v \in V\}.$$

We see that every vector in Ker(T), including the zero vector in the domain, is mapped to the zero vector in W. Considering that the kernel is exactly the null space, then it is also a subspace of $\mathbb{R}^n$. It then follows that for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the range can be represented as the column space of the matrix of T, $A = [a_1, a_2, \ldots, a_n]$, so Rng$(T)$ is a subspace of $\mathbb{R}^m$.

Example 6.12. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with matrix $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \end{bmatrix}$. We want to determine the kernel and range of T.

To determine the kernel, we need to find the solution set to $Ax = 0$. Considering $A \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ gives $x_1 = 2r - 5s$, $x_2 = r$, $x_3 = s$. Hence,

$$\text{Ker}(T) = \left\{x \in \mathbb{R}^3 : x = (2r - 5s, r, s), r, s \in \mathbb{R}\right\}.$$

Here, $\text{Ker}(T)$ is the two-dimensional subspace of $\mathbb{R}^3$ spanned by $(2, 1, 0)$ and $(-5, 0, 1)$. The linear transformation maps all points on the subspace to the zero vector in $\mathbb{R}^2$.
Since T is a matrix transformation,

$$\text{Rng}(T) = \text{colspace}(A) = \left\{y \in \mathbb{R}^2 : y = r(1, -2), r \in \mathbb{R}\right\}.$$

Now we understand T better: T maps all points in $\mathbb{R}^3$ onto the line of $\text{Rng}(T)$, a one-dimensional subspace of $\mathbb{R}^2$. Specifically, $\text{Ker}(T)$ is mapped onto the zero vector in $\mathbb{R}^2$.

The example gives us important information regarding $\text{Ker}(T)$ and $\text{Rng}(T)$:

$$\text{Ker}(t) = \text{nullspace}(A) \subset \mathbb{R}^n \quad \text{Rng}(T) = \text{colspace}(A) \subset \mathbb{R}^m$$

Theorem 6.13

If $T : V \to W$ is a linear transformation, then

(1) $\text{Ker}(T)$ is a subspace of V.

(2) $\text{Rng}(T)$ is a subspace of W.

Proof. We know that both $\text{Ker}(T)$ and $\text{Rng}(T)$ include the zero vector and are subsets of their respective vector space, establishing closeness under addition and scalar multiplication suffices.

(1) If $v_1, v_2 \in \text{Ker}(T)$, then $T(v_1) = 0$ and $T(v_2) = 0$. Consider $T(v_1 + v_2)$ and $T(cv_1)$, we have

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0 \quad T(cv_1) = cT(v_1) = 0$$

As $0 \in \text{Ker}(T)$, $\text{Ker}(T)$ is closed under addition and scalar multiplication. Thus, $\text{Ker}(T)$ is a subspace of V.

(2) If $w_1, w_2 \in \text{Rng}(T)$, then $w_1 = T(v_1)$ and $w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Thus,

$$w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2) \quad cw_1 = cT(v_1) = T(cv_1)$$

This means both $w_1 + w_2$ and cw_1 are in $\text{Rng}(T)$ as they are output of T. Hence, $\text{Rng}(T)$ is closed under addition and scalar multiplication, and it follows that $\text{Rng}(T)$ is a subspace of W.

□

Theorem 6.14: The general rank-nullity theorem

If $T : V \to W$ is a linear transformation and V is finite-dimensional, then

$$\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V]$$

Proof. Suppose that $\dim[V] = n$. We consider three cases:

Case 1. If $\dim[\text{Ker}(T)] = n$, then $\text{Ker}(T) = V$ (Goode Corollary 4.6.14: if $\dim[V] = n$ and S is a subspace of V, then if $\dim[S] = n$, $S = V$), that $T(v) = 0$ for every $v \in V$. Now we know that $\text{Rng}(T)$ only includes the zero vector, so its dimension is zero. The general R-N theorem holds.

Case 2. If $\dim[\text{Ker}(T)] = k$, where $0 < k < n$, then the basis for $\text{Ker}(T)$ is $\{v_1, v_2, \ldots, v_k\}$. We can extend the basis to a basis for V: $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n$. It suffices to prove that $T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)$ is a basis for $\text{Rng}(T)$. Consider $w \in \text{Rng}(T)$, then $w = T(v)$ for some $v \in V$. We then have $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ for some $c_1, c_2, \ldots, c_n$. Hence,

$$w = T(v) = c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n).$$

Since $v_1, v_2, \ldots, v_k \in \text{Ker}(T)$, $w = 0 + 0 + \ldots + 0 + c_{k+1}T(v_{k+1}) + c_{k+2}T(v_{k+2}) + \ldots + c_nT(v_n)$. Thus,

$$\text{Rng}(T) = \text{span}\{T(v_{k+1}, T(v_{k+2}), \ldots, T(v_n)\}.$$

Now we show that the vectors in the span is linearly independent. Suppose that

$$d_{k+1}T(v_{k+1}) + d_{k+2}T(v_{k+2}) + \ldots + d_nT(v_n) = 0,$$

then it also follows that $T(d_{k+1}v_{k+1} + d_{k+2}v_{k+2} + \ldots + d_nv_n) = 0$, hence $d_{k+1}v_{k+1} + d_{k+2}v_{k+2} \ldots + d_nv_n \in \text{Ker}(T)$. Consequently, there exists $d_1, d_2, \ldots, d_k$ such that

$$d_{k+1}v_{k+1} + d_{k+2}v_{k+2} + \ldots + d_nv_n = d_1v_1 + d_2v_2 + \ldots + d_kv_k \Rightarrow d_1v_1 + d_2v_2 + \ldots + d_kv_k - (d_{k+1}v_{k+1} + d_{k+2}v_{k+2} + \ldots + d_nv_n) = 0.$$

Here is an important catch: the set of vectors $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n$ is linearly independent as it is the basis of V. Hence, the only solution to the system is $d_1 = d_2 = \ldots = d_k = d_{k+1} = \ldots = d_n = 0$. Thus, $\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ is linearly independent, so it is a basis for $\text{Rng}(T)$, and $\dim[\text{Rng}(T)] = n - k$. It follows that

$$\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = k + (n - k) = n = \dim[V].$$

Case 3. If $\dim[\text{Ker}(T)] = 0$, then $\text{Ker}(T)$ only includes the zero vector. Then we can let $\{v_1, v_2, \ldots, v_n\}$ be any basis for V. We use a similar argument to case 2 here. Consider $w \in \text{Rng}(T)$, then $w = T(v)$ for some $v \in V$.

$$\text{Rng}(T) = \text{span}\{T(v_1), T(v_2), \ldots, T(v_n)\}.$$

Now suppose that $d_1T(v_1) + d_2T(v_2) + \ldots + d_nT(v_n) = 0$, then $T(d_1v_1 + d_2v_2 + \ldots + d_nv_n) = 0$ We already know that the set of vectors $v_1, v_2, \ldots, v_n$ is linearly independent as it is the basis of V, so similarly, $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is also linear independent, so it is a basis for $\text{Rng}(T)$, and $\dim[\text{Rng}(T)] = n$. Again, we have

$$\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = 0 + n = n = \dim[V].$$

$\square$

6.4 Additional Properties of Linear Transformations

We want to establish that all real vector spaces of a finite dimension n are essentially the same as $\mathbb{R}^n$, and we do so in this section by considering the composition of linear transformations.

Definition 6.15: Composition of linear transformations

Consider $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations. We define the **composition**, or **product**, $T_2T_1 : U \to W$ by:

$$(T_2T_1)(u) = T_2(T_1(u)) \quad \forall u \in U.$$

Theorem 6.16

Let $T_1 : U \to V$ and $T_2 : V \to W$ be linear transformations. Then $T_2T_1 : U \to W$ is a linear transformation.

Proof. Consider arbitrary vectors $u_1, u_2 \in U$, and $c \in \mathbb{R}$. It suffices to prove $(T_2T_1)(u_1 + u_2) = (T_2T_1)u_1 + (T_2T_1)u_2$ and $(T_2T_1)(cu_1) = c(T_2T_1)(u_1)$. The first equation can be proven as follows:

$$\begin{aligned}(T_2T_1)(u_1 + u_2) &= T_2(T_1(u_1 + u_2))\\ &= T_2(T_1(u_1) + T_1(u_2))\\ &= T_2(T_1(u_1)) + T_2(T_1(u_2))\\ &= (T_2T_1)(u_1) + (T_2T_1)(u_2).\end{aligned}$$

And, the second equation, as follows:

$$\begin{aligned}(T_2T_1)(cu_1) &= T_2(T_1(cu_1))\\ &= T_2(cT_1(u_1))\\ &= cT_2(T_1(u_1))\\ &= c(T_2T_1)(u_1).\end{aligned}$$

□

Note that the outputs from the linear transformation T_1 become the inputs for the linear transformation T_2 when computing the composition T_2T_1. Hence, the commutative property does *not* hold. Even when both compositions T_1T_2 and T_2T_1 make mathematical sense, they may not be the same linear transformation.

Example 6.17. Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with matrices A and B respectively. We want to determine the linear transformation $T_2T_1 : \mathbb{R}^n \to \mathbb{R}^p$.
We compute the linear transformation by directly using the definition.

$$(T_2T_1)(x) = T_2(T_1(x)) = T_2(Ax) = B(Ax) = (BA)x.$$

Consequently, T_2T_1 is the linear transformation with matrix BA. Note that A is an $m \times n$ matrix, and B is a $p \times m$ matrix. BA is then defined with size $p \times n$, which transforms $\mathbb{R}^n$ to $\mathbb{R}^p$.

Definition 6.18

A linear transformation $T : V \to W$ is said to be

(1) **one-to-one** if distinct elements in V are mapped via T to distinct elements in W. This means, $v_1 \neq v_2 \in V$ implies $T(v_1) \neq T(v_2)$.

(2) **onto** if the range of T is the entirety of W. This means, if *every* $w \in W$ is the image under T of at least one vector $v \in V$.

Theorem 6.19

Let $T : V \to W$ be a linear transformation. Then T is one-to-one if and only if $\mathrm{Ker}(T) = \{0\}$.

Proof. Since T is a linear transformation, $T(0) = 0$. If T is one-to-one, there can be no other vector $v \in V$ satisfying $T(v) = 0$, otherwise it would contradict with the definition. Hence, $\mathrm{Ker}(T) = \{0\}$.
Conversely, suppose that $\mathrm{Ker}(T) = \{0\}$. If $v_1 \neq v_2$, then $v_1 - v_2 \neq 0$, and therefore $T(v_1 - v_2) \neq 0$. The linearity property follows, giving $T(v_1) - T(v_2) \neq 0$. Hence, $T(v_1) \neq T(v_2)$, and T is one-to-one. □

Now we have reached the conclusion that the linear transformation $T : V \to W$ is one-to-one if and only if $\mathrm{Ker}(T) = \{0\}$, and onto if and only if $\mathrm{Rng}(T) = W$.

Example 6.20. Consider the transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = (2a - b + c) + (b - 2a)x + cx^2.$$

We want to determine whether T is one-to-one, onto, both, or neither.
We first test if T is one-to-one. Consider $T(a+bx+cx^2) = 0$, then it must be satisfied that $2a-b+c = b-2a = c = 0$. Computing the system gives the solution $c = 0$, $b = 2a$, so $\mathrm{Ker}(T) = \{a(1 + 2x) : a \in \mathbb{R}\}$. Hence, T is not one-to-one.
Then, we test if T is onto. As $\mathrm{Ker}(T) = \{a(1 + 2x)\}$, its dimension is one. Hence, $\dim[\mathrm{Rng}(T)] = \dim[V] - 1 = 2$, which means $\mathrm{Rng}(T)$ is a two-dimensional subspace of the three-dimensional vector space $P_2(\mathbb{R})$, so clearly T is not onto. Thus, T is neither one-to-one nor onto.

If $T : V \to W$ is both one-to-one and onto, then for each $w \in W$, there is a unique $v \in V$ such that $T(v) = w$. We can therefore define a mapping $T^{-1} : W \to V$ by

$$T^{-1}(w) = v \Leftrightarrow w = T(v).$$

Definition 6.21: Inverse transformation

Let $T : V \to W$ be a linear transformation. If T is both one-to-one and onto, then the linear transformation $T^{-1} : W \to V$ defined by

$$T^{-1}(w) = v \Leftrightarrow w = T(v)$$

is called the **inverse transformation** to T.

Theorem 6.22

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Then T^{-1} exists if and only if $\det(A) \neq 0$. Furthermore, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with matrix A^{-1}.

Proof. Consider the kernel of T. T is one-to-one if and only if $\text{Ker}(T) = \{0\}$, i.e., if and only if the linear system $Ax = 0$ has only the trivial solution, which is true if and only if $\det(A) \neq 0$. Furthermore,

$$\text{Rng}(T) = \{Ax : x \in \mathbb{R}^n\} = \text{colspace}(A).$$

Consequently, the following statements are equivalent:

(1) T is onto.

(2) $\text{colspace}(A) = \mathbb{R}^n$.

(3) The column vectors of A span $\mathbb{R}^n$.

(4) $\det(A) \neq 0$.

□

Finally, if $\det(A) \neq 0$, then A^{-1} exists, so that $T(x) = y \Leftrightarrow Ax = y \Leftrightarrow x = A^{-1}y$. Thence, $T^{-1}(y) = A^{-1}y$, from which it follows that T^{-1} is itself a linear transformation with matrix A^{-1}.

Definition 6.23: Isomorphism

Let V and W be vector spaces. If there exists a linear transformation $T : V \to W$ that is both one-to-one and onto, we call T an **isomorphism**, and we say that V and W are isomorphic vector spaces, $V \cong W$.

Example 6.24. We want to determine an isomorphism $T : \mathbb{R}^3 \to P_2(\mathbb{R})$.
We do this by considering an arbitrary vector in $P_2(\mathbb{R})$, expressed as $a_0 + a_1x + a_2x^2$. Consequently, an isomorphism between $\mathbb{R}^3$ and $P_2(R)$ can be defined by $T(a_0, a_1, a_2) = a_0 + a_1x + a_2x^2$.

Theorem 6.25

Let A be an $n \times n$ matrix with real elements, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the matrix transformation defined by $T(x) = Ax$. The following conditions are equivalent:

(1) A is invertible.

(2) T is one-to-one.

(3) T is onto.

(4) T is an isomorphism.

Proof. The proof is trivial. (1) and (2) by IMT, (2) and (3) by Goode Prop 4.6.14, (3) and (4) by definition. □

6.5 The Matrix of a Linear Transformation

Definition 6.26: Matrix representation

Let V and W be vector spaces with ordered bases $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_m\}$, respectively, and let $T : V \to W$ be a linear transformation. The $m \times n$ matrix

$$[T]_B^C = [[T(v_1)]_C, [T(v_2)]_C, \ldots, [T(v_n)]_C]$$

is called the **matrix representation of T relative to the bases B and C**. For $V = W$ and $B = C$, we refer to $[T]_B^B$ as the **matrix representation of T relative to the basis B**.

Example 6.27. Consider the linear transformation $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T(a + bx) = (2a - 3b) + (b - 5a)x + (a + b)x^2.$$

We want to determine the matrix representation of T relative to the bases $B = \{1, x\}$ and $C = \left\{1, x, x^2\right\}$. We first have $T(1) = 2 - 5x + x^2$ and $T(x) = -3 + x + x^2$. So

$$[T(1)]_C = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad [T(x)]_C = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow [T]_B^C = \begin{bmatrix} 2 & -3 \\ -5 & 1 \\ 1 & 1 \end{bmatrix}.$$

Theorem 6.28

Let V and W be vector spaces with ordered bases B and C, respectively. If $T : V \to W$ is a linear transformation and v is any vector in V, then

$$[T(v)]_C = [T]_B^C [v]_B.$$

Proof. Let $B = \{v_1, v_2, \ldots, v_n\}$, and consider $v \in V$. As $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$,

$$[T]_B^C [v]_B = a_1[T(v_1)]_C + a_2[T(v_2)]_C + \ldots + a_n[T(v_n)]_C = [T(v)]_C.$$

□

We now consider the composition of linear transformations and how they can be represented with matrix representations with respect to respective bases.

Theorem 6.29

If U, V, W are vector spaces with ordered bases A, B, and C, and $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then

$$[T_2 T_1]_A^C = [T_2]_B^C [T_1]_A^B.$$

Proof. It suffices to show that premultiplying any column vector $[u]_A$ gives the same result. Here,

$$[T_2]_B^C [T_1]_A^B [u]_A = [T_2]_B^C [T_1(u)]_B = [T_2(T_1(u))]_C = [(T_2 T_1)u]_C = [T_2 T_1]_A^C [u]_A.$$

□

Now, because of the close relationship between the matrix representation of linear transformation and the linear transformation itself, the following theorem can be established.

Theorem 6.30

Let $T : V \to W$ be a linear transformation, and let B and C be ordered bases for V and W, respectively. Then

(1) For all $v \in V$, $v \in \text{Ker}(T)$ if and only if $[v]_B \in \text{nullspace}([T])_B^C$.

(2) For all $w \in W$, $w \in \text{Rng}(T)$ if and only if $[w]_C \in \text{colspace}([T]_B^C)$.

Corollary 6.31

Let $T : V \to W$ be a linear transformation, and let B and C be ordered bases for V and W respectively. Then

(1) T is one-to-one if and only if $\text{nullspace}([T]_B^C) = \{0\}$.

(2) T is onto if and only if $\text{colspace}([T]_B^C = \mathbb{R}^n$, where $n = \dim[W]$.

Given an invertible linear transformation T with matrix representation $[T]_B^C$, we can now use matrices to determine the inverse linear transformation T^{-1}. As $[T^{-1}]_C^B[T]_B^C = [I]_B^B$ and $[T]_B^C[T^{-1}]_C^B = [I]_C^C$, we can get that $\left([T]_B^C\right)^{-1} = [T^{-1}]_C^B$.

Example 6.32. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined via

$$T(a + bx + cx^2) = (3a - b + c) + (a - c)x + (4b + c)x^2.$$

(a) Find the matrix representation of T relative to the standard basis $B = \{1, x, x^2\}$ on $P_2(\mathbb{R})$.

We do this problem by considering $T(1) = 3 + x$, $T(x) = -1 + 4x^2$, and $T(x^2) = 1 - x + x^2$. Hence,

$$[T]_B^B = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 4 & 1 \end{bmatrix}.$$

(b) Use the matrix in part (a) to prove that T is invertible.

$\det[T]_B^B = -12 + 1 - 4 = -15 \neq 0$. Hence, $[T]_B^B$ is invertible, so T is invertible.

(c) Determine the linear transformation $T^{-1} : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by using the matrix representation of T^{-1} relative to $B = \{1, x, x^2\}$.

$$[T^{-1}]_B^B = \left([T]_B^B\right)^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 4 & 1 \end{bmatrix}^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 5 & 1 \\ -1 & 3 & 4 \\ 4 & -12 & 1 \end{bmatrix}.$$

Thus, $T^{-1}(a + bx + cx^2) = \frac{1}{17}[(4a + 5b + c) + (-a + 3b + 4c)x + (4a - 12b + c)x^2]$.

7 Eigenvalues and Eigenvectors

7.1 The Eigenvalue/Eigenvector Problem

Definition 7.1: Eigenvalue and eigenvector

Let A be an $n \times n$ matrix. Any values of λ for which

$$Av = \lambda v$$

has nontrivial solutions v are called the **eigenvalues** of A. The corresponding nonzero vectors v are called **eigenvectors** of A. (They are also referred to as characteristic values and characteristic vectors of A.)

Consider A as the matrix of a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$. We restrict our attention to real A and λ, focusing only on $\mathbb{R}^n$.

The eigenvectors of A are nonzero vectors that are mapped into a constant scalar multiple of themselves by T. Geometrically, the linear transformation leaves the direction of v unchanged.

Note that if $Av = \lambda v$ and c is a scalar, then

$$A(cv) = cAv = c(\lambda v) = \lambda(cv).$$

Consequently, if v is an eigenvector of A, then so is cv for any nonzero scalar c.

The solution of the eigenvalue/eigenvector problem is equivalent to solving

$$(A - \lambda I)v = 0$$

The eigenvalues of A are those values of λ for which the $n \times n$ linear system has nontrivial solutions, and the eigenvectors are the corresponding solutions.

Hence, the eigenvalue/eigenvector problem can be solved as follows:

(1) Find all scalars λ with $\det(A - \lambda I) = 0$. These are the eigenvalues of A.

(2) If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues in (1), then solving k systems of linear equations

$$(A - \lambda_i I)v_i = 0$$

to find all eigenvectors v_i corresponding to each eigenvalue.

Definition 7.2: Characteristic polynomial and characteristic equation

For a given $n \times n$ matrix A, the polynomial $p(\lambda)$ defined by

$$p(\lambda) = \det(A - \lambda I)$$

is called the characteristic polynomial of A, and the equation

$$p(\lambda) = 0$$

is called the characteristic equation of A.

Example 7.3. Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

The linear system is $(A - \lambda I)v = 0$, so we have

$$\begin{bmatrix} 3-\lambda & -1 \\ -5 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The system has nontrivial solutions if $(3-\lambda)(-1-\lambda) - 5 = 0$, giving $\lambda = \{-2, 4\}$.

Eigenvalue $\boldsymbol{\lambda_1 = -2}$: $(A - \lambda_1 I)v = 0 \Rightarrow \begin{bmatrix} 5 & -1 & 0 \\ -5 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The solution is then $v = r(1,5)$.

Eigenvalue $\boldsymbol{\lambda_2 = 4}$: $(A - \lambda_2 I)v = 0 \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ -5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The solution is then $v = s(-1,1)$.

Theorem 7.4

Let A be an $n \times n$ matrix with real elements. If λ is a complex eigenvalue of A with corresponding eigenvector v, then $\overline{\lambda}$ is the eigenvalue of A with corresponding eigenvector $\overline{v}$.

Proof. If $Av = \lambda v$, then $\overline{Av} = \overline{\lambda v}$, which implies $A\overline{v} = \overline{\lambda}\overline{v}$, since A has real entries. □

7.2 General Results for Eigenvalues and Eigenvectors

For a given $n \times n$ matrix $A = [a_{ij}]$, the characteristic polynomial $p(\lambda)$ assumes the form

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}.$$

Expanding the determinant yields a polynomial of degree n in λ with leading coefficient $(-1)^n$. It follows that $p(\lambda) = (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_k)^{m_k}$, so $m_1 + m_2 + \dots + m_k = n$ by the fundamental theorem of algebra. Thus, associated with each eigenvalue λ_i is a number m_i, called the **multiplicity** of λ_i.

Definition 7.5: Eigenspace

Let A be an $n \times n$ matrix. For a given eigenvalue λ_i, let E_i denote the set of all vectors v satisfying $Av = \lambda_i v$. Then E_i is called the **eigenspace** of A corresponding to the eigenvalue λ_i. E_i is the solution set to the linear system $(A - \lambda_i I)v = 0$.

Example 7.6. We want to determine the eigenspaces for the matrix $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$. We do this by considering the eigenvectors $v = r(1,5)$, $v = s(-1,1)$. Then we have

$$E = \{v \in \mathbb{R} : v = r(1,5) \cup s(-1,1), r, s \in \mathbb{R}\}.$$

From the above example, we have one main result for eigenspaces. Let λ_i be an eigenvalue of A of multiplicity m_i and let E_i denote the corresponding eigenspace. Then for each i, E_i is a subspace of $\mathbb{C}^n$, and the dimension of the eigenspace corresponding to λ_i is at most the multiplicity of λ_i.

Example 7.7. We want to determine all eigenspaces and their dimensions for the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$.

The characteristic polynomial is $p(\lambda) = -(\lambda-2)^2(\lambda-3)$, giving $\lambda_1 = 2$, $\lambda_2 = 3$.

For $\lambda_1 = 2$, we have $A^{\#} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$, which gives the general solution system $v = r(1,1,0) + s(0,0,1)$, so $E_1 = \left\{v \in \mathbb{R}^3 : v = r(1,1,0) + s(0,0,1), r, s \in \mathbb{R}\right\}$. Hence, we have $\dim[E_1] = 2$, so $n_1 = 2$.

For $\lambda_2 = 3$, we have $A^{\#} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$, which gives the general solution system $v = t(1,0,-1)$, so $E_2 = \left\{v \in \mathbb{R}^3 : v = t(1,0,-1), t \in \mathbb{R}\right\}$. Hence, we have $\dim[E_2] = 1$, so $n_2 = 1$.

Theorem 7.8

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Proof. We use induction to prove the result. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $v_1, v_2, \ldots, v_m$. It is true that $\{v_1\}$ is linearly independent. Now suppose $\{v_1, v_2, \ldots, v_k\}$ is linearly independent for some $k < m$. Consider the set $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$, we consider

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k + c_{k+1} v_{k+1} = 0.$$

Premultiplying both sides by A gives

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_k \lambda_k v_k + c_{k+1} \lambda_{k+1} v_{k+1} = 0.$$

As $c_{k+1} v_{k+1} = -(c_1 v_1 + c_2 v_2 + \ldots + c_k v_k)$, we rewrite the equation as

$$c_1 \lambda_1 v_1 + c_2 \lambda v_2 + \ldots + c_k \lambda_k v_k - \lambda_{k+1}(c_1 v_1 + c_2 v_2 + \ldots + c_k v_k) = 0 \Rightarrow c_1(\lambda_1 - \lambda_{k+1}) v_1 + \ldots + c_k(\lambda_k - \lambda_{k+1}) v_k = 0.$$

Since $v_1, v_2, \ldots, v_k$ are linearly independent, this implies that $c_i(\lambda_i - \lambda_{k+1}) = 0$ for $1 \leqslant i \leqslant k$. As the eigenvalues are distinct, the only solution is $c_1 = c_2 = \ldots = c_k = 0$, so that the vectors are linearly independent. □

Definition 7.9: Nondefective matrices

An $n \times n$ matrix A that has n linearly independent eigenvectors is called **nondefective**. We say that A has a complete set of eigenvectors. If A has less than n linearly independent eigenvectors, it is called **defective**.

If A is nondefective, then any set of n linearly independent eigenvectors of A is a basis for $\mathbb{R}^n$. Such a basis is referred to as an **eigenbasis** of A.

Example 7.10. We want to determine whether $A = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$ is defective or not.

The characteristic polynomial of A is $p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$.
Thus, $\lambda_1 = -2$ is an eigenvalue of multiplicity 2. The eigenvectors of A have the form $v = r(-1, 1)$. Here, we see that the eigenspace is spanned by one vector, thus $\dim[E_1] = 1 < 2$, and hence A is defective.

Theorem 7.11

An $n \times n$ matrix A is nondefective if and only if the dimension of each eigenspace is the same as the algebraic multiplicity m_i of the corresponding digenvalue.

Proof. Suppose that A is nondefective, with eigenspaces $E_1, E_2, \ldots, E_k$ of dimensions $n_1, n_2, \ldots, n_k$, respectively. Since A is nondefective, $n_1 + n_2 + \ldots + n_k = n$. If $n_i < m_i$ for some i, then we have

$$n = n_1 + n_2 + \ldots + n_k < m_1 + m_2 + \ldots + m_k = n,$$

a clear contradiction. Thus, $n_i = m_i$ for each i, which is equivalent to the statement that the dimension of each eigenspace is the same as the algebraic multiplicity of the eigenvalue.
Conversely, if $n_i = m_i$ for each i, then

$$n = m_1 + m_2 + \ldots + m_k = n_1 + n_2 + \ldots + n_k,$$

which means that the union of the linearly independent eigenvectors that span each eigenspace consists of n eigenvectors of A, and this union is linearly independent. Thus, A has n linearly independent eigenvectors. $\square$

7.3 Diagonalization

Consider the linear system of differential equations

$$dx_1/dt = a_{11}x_1 + a_{12}x_2, \quad dx_2/dt = a_{21}x_1 + a_{22}x_2.$$

We can then write this as a vector equation $x' = Ax$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \quad A = \begin{bmatrix} a_{ij} \end{bmatrix}$$

Suppose we make a linear change of variables defined by $x = Sy$, where S is an invertible matrix. Then,

$$x' = Sy' \Rightarrow Sy' = ASy.$$

Premutiplying by S^{-1} yields $y' = By$, where $B = S^{-1}AS$. The question is whether it is possible to choose S such that $y' = By$ can be integrated. We then lead to the definition of similar matrices.

Definition 7.12: Similar matrices

Let A and B be $n \times n$ matrices. We say A is **similar to** B if there exists an invertible matrix S such that $B = S^{-1}AS$.

Example 7.13. If $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 22 & 6 \\ -70 & -19 \end{bmatrix}$, we want to verify that $B = S^{-1}AS$, where $S = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$.
It is obvious that $S^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$, so $S^{-1}AS = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 22 & 6 \\ -70 & -19 \end{bmatrix} = B$.

Theorem 7.14

Similar matrices have the same eigenvalues.

Proof. If A is similar to B, then $B = S^{-1}AS$ for some invertible matrix S. Thus,

$$\begin{aligned} \det(B - \lambda I) &= \det(S^{-1}AS - \lambda I) = \det(S^{-1}AS - \lambda S^{-1}S) \\ &= \det(S^{-1}(A - \lambda I)S) = \det(S^{-1})\det(A - \lambda I)\det(S) \\ &= \det(A - \lambda I). \end{aligned}$$

□

We see that A and $B = S^{-1}AS$ have the same eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Furthermore, we also know that the simplest possible matrix that has these eigenvalues is $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. This leads to the question: for an $n \times n$ matrix A, when does an invertible matrix S exist such that $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$?
We provide the answer in the next theorem.

Theorem 7.15

An $n \times n$ matrix A is similar to a diagonal matrix if and only if A is nondefective. In such a case, if $v_1, v_2, \ldots, v_n$ denote n linearly independent eigenvectors of A and $S = \left[v_1, v_2, \ldots, v_n\right]$, then

$$S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A corresponding to the eigenvectors $v_1, v_2, \ldots, v_n$.

Proof. If A is similar to a diagonal matrix, then there exists an invertible matrix $S = \left[v_1, v_2, \ldots, v_n\right]$ such that $S^{-1}AS = D$, where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. From theorem 7.3.3, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. Premultiplying both sides by S gives $AS = SD$, or equivalently,

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2, \quad \ldots, \quad Av_n = \lambda_n v_n.$$

Consequently, $v_1, v_2, \ldots, v_n$ are eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Further, since $\det(S) \neq 0$, the eigenvectors are linearly independent.
Conversely, suppose A is nondefective, and let $S = \left[v_1, v_2, \ldots, v_n\right]$, where $\{v_1, v_2, \ldots, v_n\}$ is any complete set of eigenvectors of A. Then

$$AS = A\left[v_1, v_2, \ldots, v_n\right] = [Av_1, Av_2, \ldots, Av_n] = \left[\lambda_1 v_1, \lambda_2 v_2, \ldots., \lambda_n v_n\right].$$

This is equivalent to $AS = SD$, where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Since the columns of S form a linearly independent set, $\det(S) \neq 0$, and hence S is invertible. Premultiplying both sides of $AS = SD$ by S^{-1} gives

$$S^{-1}AS = D.$$

□

Definition 7.16: Diagonalizable matrix

An $n \times n$ matrix that is similar to a diagonal matrix is said to be **diagonalizable**.

Example 7.17. We want to determine all solutions to $x_1' = 9x_1 + 6x_2$, $x_2' = -10x_1 - 7x_2$. Consider $x' = Ax$, where $A = \begin{bmatrix} 9 & 6 \\ -10 & -7 \end{bmatrix}$. The transformed system is $y' = (S^{-1}AS)y$, where $x = Sy$. To determine S, we need the eigenvalues and eigenvectors of A. The characteristic polynomial is $p(\lambda) = (\lambda - 3)(\lambda + 1)$. From this we know that A is nondefective, $v = r(-1, 1)$ and $v = s(-3, 5)$.
Setting $S = \begin{bmatrix} -1 & -3 \\ 1 & 5 \end{bmatrix}$ gives $S^{-1}AS = \text{diag}(3, -1)$, so that the system is $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.
From this, we get $y_1' = 3y_1$, $y_2' = -y_2$, which gives $y_1(y) = c_1 e^{3t}$, $y_2(t) = c_2 e^{-t}$.
Consequently, $x_1(t) = -c_1 e^{3t} - 3c_2 e^{-t}$ and $x_2(t) = c_1 e^{3t} + 5c_2 e^{t}$.

8 Linear Differential Equations of Order n

In Chapter 1, we developed techniques that enabled us to solve first-order linear differential equations. However, there are lots of equations that are of orders greater than one, so those techniques will not be as useful when dealing with, say, the RLC circuit:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Recall that any such differential equation can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $a_0, a_1, \ldots, a_n, F$ are functions defined an interval I. In this chapter, we will apply the results from chapters 4 and 6 to develop a theory for the solution of such equations, primarily through three steps:

(1) Reformulate the problem in the equivalent form

$$Ly = F,$$

where L is an appropriate linear transformation.

(2) Establish that the set of all solutions to the associated homogeneous differential equation

$$Ly = 0$$

is a vector space of dimension n, so that every solution to the homogeneous differential equation can expressed as

$$y(x) = c_1y(x) + c_2y_2(x) + \ldots + c_ny_n(x),$$

where $\{y_1, y_2, \ldots, y_n\}$ is any linearly independent set of n solutions to $Ly = 0$.

(3) Establish that every solution to the nonhomogeneous problem $Ly = F$ is of the form

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x),$$

where $y_p(x)$ is any particular solution to the nonhomogeneous equation.

8.1 General Theory for Linear Differential Equations

Recall that $D : C^1(I) \to C^0(I)$ defined by $D(f) = f'$ is a linear transformation. We call D the **derivative operator**. Higher-order derivative operators have $D^k : C^k(I) \to C^0(I)$ defined by

$$D^k = D(D^{k-1}), \quad k = 2, 3, \ldots \Rightarrow D^k(f) = \frac{d^kf}{dx^k}$$

Taking a linear combination of the basic derivative operators, the general linear differential operator of order n is:

$$L = D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n,$$

and is defined by

$$Ly = y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny.$$

Now consider the general n-th order linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $a_0, a_1, \ldots, a_n$ and F are functions specified on an interval I. If $F(x)$ is identically zero on I, then the differential equation is **homogeneous**. Otherwise, it is **nonhomogeneous**. Assuming that $a_0(x)$ is nonzero on I, we can divide the differential equation by a_0 to obtain the following standard form:

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = F(x) \Rightarrow Ly = F(x).$$

Theorem 8.1: General existence and uniqueness theorem

Let $a_1, a_2, \ldots, a_n$ and F be functions that are continuous on an interval I. Then, for any $x_0 \in I$, the initial-value problem

$$Ly = F(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

has a unique solution on I.

We don't formalize the proof for the general existence and uniqueness theorem here, as it requires concepts from calculus, namely the Lebesgue's Dominated Convergence Theorem.
The differential equation is said to be **regular** on I if the functions $a_1, a_2, \ldots, a_n, F$ are continuous on I. For now, we always assume that the differential equations are regular on the interval of interest.

We first consider the n-th order linear homogeneous differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = 0$$

on an interval I. This can be written as the operator equation $Ly = 0$, where $L : C^n(I) \to C_0(I)$ is defined by

$$L = D^n + a_1 D^{n-1} + \ldots + a_{n-1}D + a_n.$$

Let the set of all solutions be S, it is clear that

$$S = \{y \in C^n(I) : Ly = 0\} = \text{Ker}(L).$$

As the kernel of any linear transformation $T : V \to W$ is a subspace of V, the set of all solutions to the differential equation is a subspace of $C^n(I)$. This subspace is referred to as the **solution space** of the differential equation. If we can determine the dimension of S, then we will know how many linearly independent solutions are required to span the solution space.

Theorem 8.2

The set of all solutions to the regular n-th order homogeneous linear differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = 0$$

on an interval I is a vector space of dimension n.

Proof. Consider the operator form $Ly = 0$. To prove the dimension of the solution space is n, we must establish the existence of a basis consisting of n solutions.

Let $y_1, y_2, ..., y_n$ be the n solutions satisfying the initial value problems

$$Ly_1 = 0, \quad y_1(x_0) = 1, \quad y_1'(x_0) = ... + y^{(n-1)}(x_0) = 0$$

$$Ly_2 = 0, \quad y_2'(x_0) = 1, \quad y_2(x_0) = ... + y^{(n-1)}(x_0) = 0$$

$$...$$

$$Ly_n = 0, \quad y_n^{(n-1)}(x_0) = 0, \quad y_n(x_0) = ... = y^{(n-2)}(x_0) = 0$$

Consider the Wronskian of these solutions: $W = 1 \neq 0$, implying that all n solutions are linearly independent. In addition to that, L is indeed a linear transformation satisfying

$$L(ky) = kL(y) \quad L(y_1 + y_2) = L(y_1) + L(y_2),$$

Hence, any solution to $Ly = 0$ can be written as a linear combination of the linearly independent solutions $y_1, ..., y_n$, thus these solutions span the solution space. The proof follows immediately. □

It follows from the previous theorem that any set of n linearly independent solutions $\{y_1, y_2, ..., y_n\}$ to

$$y^{(n)} + a_1(x)y^{(n-1)} + ... + a_{n-1}(x)y' + a_n(x)y = 0$$

is a basis for the solution space of this differential equation. Consequently, every solution can be written as

$$y(x) = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x),$$

where $c_1, c_2, ..., c_n$ are constants. $y(x)$ is referred to as the **general solution** to the differential equation.

Example 8.3. We want to determine all solutions to the differential equation $y'' - 2y' - 15y = 0$ of the form $y(x) = e^{rx}$, where r is a constant, and using this to determine the general solution.
We first consider $y(x) = e^{rx}$, then $y'(x) = re^{rx}$ and $y''(x) = r^2 e^{rx}$. Substitution gives

$$e^{rx}(r^2 - 2r - 15) = 0 \Rightarrow (r+3)(r-5) = 0$$

Hence, the two solutions to the differential equation are

$$y_1(x) = e^{-3x} \quad y_2(x) = e^{5x}.$$

Furthermore, $W = 8e^{2x} \neq 0$, so that $y_1(x)$ and $y_2(x)$ are linearly independent on any interval. It follows that the set of all solutions to the differential equation is $\left\{e^{-3x}, e^{5x}\right\}$ and the general solution is

$$y(x) = c_1 e^{-3x} + c_2 e^{5x}.$$

Note that $W \neq 0$ for all ranges of x. If $W = 0$ at some point of x, in section 4.5 we concluded that we wouldn't be able to draw any conclusion as to the linear dependence or linear independence of the solutions. We now show, however, that the solutions are, in fact, linearly dependent.

Theorem 8.4

Let $y_1, y_2, \ldots, y_n$ be solutions to the regular n-th order differential equation $Ly = 0$ on an interval I, and let $W[y_1, y_2, \ldots, y_n](x)$ denote their Wronskian. If $W = 0$ at some point $x_0 \in I$, then $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on I.

Proof. Consider

$$W = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Assuming $W = 0$, the system has a nontrivial solution $(\alpha_1, \ldots, \alpha_n)$. We can now define $u(x)$ as

$$u(x) = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n.$$

Then, $y = u(x)$ satisfies the initial-value problem

$$Ly = 0 \quad y(x_0) = y'(x_0) = \ldots + y^{(n-1)}(x_0) = 0.$$

Consider the solution $y(x) = 0$, and by the existence and uniqueness theorem, we have

$$u(x) = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n = 0.$$

Then, it naturally follows that $\{y_1, y_2, \ldots, y_n\}$ must be linearly dependent. □

An important conclusion from the above theorem is that the Wronskian on an interval I completely characterizes whether solutions to $Ly = 0$ are linearly dependent or linearly independent on I.

Example 8.5. Consider $y'' + 4y = 0$. We want to verify that $y_1(x) = \cos 2x$ and $y_2(x) = 3(1 - 2\sin^2 x)$ are solutions for the differential equation on $(-\infty, \infty)$ and we want to show that they are linearly dependent. We first verify by direct substitution that

$$y_1'' + 4y_1 = 0 \quad y_2'' + 4y_2 = 0.$$

To show linear dependency, we compute their Wronskian as follows:

$$W[y_1, y_2](x) = \begin{vmatrix} \cos 2x & 3(1 - 2\sin^2 x) \\ -2\sin 2x & -12\sin x \cos x \end{vmatrix} = -6\cos 2x \sin 2x + 6\sin 2x(1 - 2\sin^2 x) = 0.$$

$W = 0$, so the solutions are linear dependent. In fact, considering $\cos 2x = 1 - 2\sin^2 x$, it is easy to see that $y_2(x) = 3y_1(x)$, and the second linearly independent solution to the differential equation is $y_3(x) = \sin 2x$.

We now consider the nonhomogeneous linear differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $F(x)$ is not identically zero on the interval of interest. If we set $F(x) = 0$, we obtain the associated homogeneous equation. In operator form, the nonhomogeneous and homogeneous equations have the forms

$$Ly = F, \quad Ly = 0.$$

Theorem 8.6

Let $\{y_1, y_2, \ldots, y_n\}$ be a linearly independent set of solutions to $Ly = 0$ on an interval I, and let $y = y_p$ be any particular solution to $Ly = F$ on I. Then every solution to $Ly = F$ on I is of the form

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n + y_p.$$

Proof. Since $y = y_p$ satisfies nonhomogeneous equation, we have

$$Ly_p = F.$$

Let $y = u$ be any solution to the differential equation, we also have $Lu = F$, which, combining with $Ly_p = F$ and applying the linearity of L, gives

$$L(u - y_p) = 0.$$

Thus, $y = u - y_p$ is a solution to the associated homogeneous equation and therefore can be written as

$$y - u_p = c_1y_1 + c_2y_2 + \ldots + c_ny_n \Rightarrow u = c_1y_1 + c_2y_2 + \ldots + c_ny_n + y_p.$$

□

Hence, the general solution to the nonhomogeneous differential equation is

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$. We refer to y_c as the **complementary function** for $Ly = F$.

Example 8.7. We want to first verify that $y_p(x) = 2e^{6x}$ is a particular solution to the differential equation

$$y'' - 2y' - 15y = 18e^{6x}$$

and use that to determine the general solution. To do so, we first verify by direct substitution. Consider example 8.3, we have $y_c(x) = c_1 e^{-3x} + c_2 e^{5x}$, giving

$$y(x) = c_1 e^{-3x} + c_2 e^{5x} + 2e^{6x}.$$

We conclude our section with a simple theorem.

Theorem 8.8

If $y = u_p$ and $y = v_p$ are particular solutions to $Ly = f(x)$ and $Ly = g(x)$, then $y = u_p + v_p$ is a solution to

$$Ly = f(x) + g(x).$$

Proof.

$$L(u_p + v_p) = L(u_p) + L(v_p) = f(x) + g(x).$$

□

8.2 Constant Coefficient Homogeneous Linear Differential Equations

In the next few sections we develop techniques for solving differential equations of order n that have constant coefficients. These are differential equations with the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = F(x),$$

where $a_1, a_2, \ldots, a_n$ are constants. To determine the general solution to this differential equation, we gin by analyzing the associated homogeneous equation

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0.$$

Here, we consider the **polynomial differential operator** $P(D)$:

$$P(D) = D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n.$$

For the homogeneous case, $P(D)y = 0$. Associated with any polynomial differential operator is the real polynomial

$$P(r) = r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n,$$

often referred to as the **auxiliary polynomial**. The corresponding polynomial equation

$$P(r) = 0$$

is called the **auxiliary equation**.

Example 8.9. We want to write the differential equation $y'' + 5y' - 7y = 0$ as $P(D)y = 0$ for an appropriate polynomial differential operator $P(D)$. Then we want to determine the auxiliary polynomial and the auxiliary equation. We do this by considering

$$(D^2 + 5D - 7)y = 0 \equiv P(D)y = 0 \Rightarrow P(D) = D^2 + 5D - 7.$$

Consequently, the auxiliary polynomial is $P(r) = r^2 + 5r - 7$ and the equation is $r^2 + 5r - 7 = 0$.

In general, the composition of two linear transformations is not commutative. However, even though $L_1 L_2 \neq L_2 L_1$, commutativity does hold for polynomial differential operators.

Theorem 8.10

If $P(D)$ and $Q(D)$ are polynomial differential operators, then

$$P(D)Q(D) = Q(D)P(D).$$

Proof. The proof is a direct verification, so is omitted here. □

The commutativity of polynomial differential operators enables the factoring of polynomial differential factors. Specifically, if $P(D)$ is a polynomial differential operator of degree n, the auxiliary polynomial can be factored as

$$P(r) = (r - r_1)^{m_1}(r - r_2)^{m_2}...(r - r_k)^{m_k},$$

where m_i is the multiplicity of the root r_i, and $m_1 + m_2 + ... + m_k = n$. Consequently, we have

$$P(D) = (D - r_1)^{m_1}(D - r_2)^{m_2}...(D - r_k)^{m_k}.$$

It follows that the differential equation $P(D)y = 0$ can be written as

$$(D - r_1)^{m_1}(D - r_2)^{m_2}...(D - r_k)^{m_k} y = 0.$$

Theorem 8.11

If $P(D) = P_1(D)P_2(D)...P_k(D)$, then for each i, the solution to $P_i(D)y = 0$ is also a solution to $P(D)y = 0$.

Proof. Suppose $P_i(D)u = 0$ for some i. Then, we have

$$P(D) = P_1(D)...P_{i-1}(D)P_{i+1}(D)...P_k(D)P_i(D),$$

hence

$$P(D)u = P_1(D)...P_{i-1}(D)P_{i+1}(D)...P_k(D)P_i(D)u = 0.$$

□

Therefore, we see that any solutions to

$$(D - r_i)^{m_i} y = 0$$

will also be solutions to the differential equation. Our next step is to find the solutions to the equation above.

Theorem 8.12

Lemma 8.12.1

Consider the differential operator $(D-r)^m$, where m is a positive integer, and r is a complex number. For any $u \in C^m(I)$,

$$(D-r)^m(e^{rx}u) = e^{rx}D^m(u).$$

Proof. When $m = 1$, we have

$$(D-r)(e^{rx}u) = e^{rx}u' + re^{rx}u - re^{rx}u = e^{rx}u'.$$

Repeating this procedure yields similar result. The proof follows by induction. □

The differential equation, where m is a positive integer and r is a complex number, has the following n solutions that are linearly independent on any interval:

$$e^{rx}, xe^{rx}, ..., x^{m-1}e^{rx}.$$

Proof. Since

$$D^m(x^k) = 0,$$

the lemma with $u(x) = x^k$ yields

$$(D-r)^m(e^{rx}x^k) = e^{rx}D^m(x^k) = 0,$$

and hence, $e^{rx}, xe^{rx}, ..., x^{m-1}e^{rx}$ are solutions to the differential equation $(D-r)^m y = 0$. Now we consider the solution e^{rx}, there is

$$c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx} + ... + c_mx^{m-1}e^{rx} = 0,$$

for x in any interval if and only if $c_1 = c_2 = ... = c_m = 0$. Dividing by e^{rx} gives

$$c_1 + c_2x + ... + c_mx^{m-1} = 0.$$

Since $\{1, x, x^2, ..., x^{m-1}\}$ is linearly independent on any interval, it follows that

$$c_1 = c_2 = ... = c_m = 0.$$

It follows that the given functions are indeed linearly independent on any interval. □

We now apply the results of the above theorems to the differential equation

$$(D-r_1)^{m_1}(D-r_2)^{m_2}...(D-r_k)^{m_k}y = 0.$$

The solutions that are obtained due to a term of the form $(D-r)^m$ depend on whether r is real or complex.

Case 1. Consider the situation where r is real. Each factor of the form $(D-r)^m$ contributes the n linearly independent solutions

$$e^{rx}, xe^{rx}, ..., x^{m-1}e^{rx}.$$

Case 2. Consider the situation where r is imaginary. Since complex roots occur in conjugate pairs, each factor of the form $(D - r)^m$ must be accompanied by a term $D - \overline{r})^m$. These complex conjugate terms contribute the complex-valued solutions

$$e^{(a \pm ib)x}, xe^{(a \pm ib)x}, \ldots, x^{m-1}e^{(a \pm ib)x}.$$

Consider the two complex conjugate solutions

$$w_1(x) = x^k e^{(a+ib)x} = x^k e^{ax}(\cos bx + i \sin bx) \quad w_2(x) = x^k e^{(a-ib)x} = x^k e^{ax}(\cos bx - i \sin bx).$$

Since these are both solutions to a linear homogeneous equation, any linear combination of them is also a solution to the same equation. In particular, consider

$$y_1(x) = \frac{1}{2}[w_1(x) + w_2(x)] = x^k e^{ax} \cos bx \quad y_2(x) = \frac{1}{2i}[w_1(x) - w_2(x)] = x^k e^{ax} \sin bx.$$

y_1 and y_2 are two corresponding real valued solutions. Repeating the process gives the set of real solutions:

$$e^{ax} \cos bx, e^{ax} \sin bx, xe^{ax} \cos bx, xe^{ax} \sin bx, \ldots, x^{m-1}e^{ax} \cos bx, x^{m-1}e^{ax} \sin bx.$$

We now summarize our results.

Theorem 8.13

Consider the differential equation $P(D)y = 0$. Let $r_1, \ldots, r_k$ be the distinct roots of the auxiliary equation, so that

$$P(r) = (r - r_1)^{m_1}(r - r_2)^{m_2} \ldots (r - r_k)^{m_k},$$

where m_i denotes the multiplicity of the root $r = r_i$.
If r_i is real, then the functions $e^{r_i x}, xe^{r_i x}, \ldots, x^{m_i - 1}e^{r_i x}$ are linearly independent solutions to the equation on any interval. If r_j is complex, consider $r_j = a + ib$, then the functions $e^{ax} \cos bx, xe^{ax} \cos bx, \ldots, x^{m_j-1}e^{ax} \cos bx, e^{ax} \sin bx, xe^{ax} \sin bx, \ldots, x^{m_j-1}e^{ax} \sin bx$ corresponding to the conjugate roots are linearly independent solutions on any interval.

The n real-valued solutions that are obtained by considering the distinct roots $r_1, r_2, \ldots, r_k$ are linearly independent on any interval. Consequently, the general solution will then be

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x).$$

Example 8.14. We want to determine the general solution to $y'' - y' - 2y = 0$. To do so, we consider the auxiliary polynomial $P(r) = r^2 - r - 2 = (r - 2)(r + 1)$. Therefore, the auxiliary equation has roots $r_1 = 2$, $r_2 = -1$. It follows that the two linearly independent solutions are

$$y_1(x) = e^{2x} \quad y_2(x) = e^{-x}.$$

Hence, the general solution to the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$

Example 8.15. We want to solve the initial-value problem $y'' + 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 4$. We first see that the auxiliary equation $r^2 + 4r + 4$ has the general solution

$$y(x) = c_1 e^{-2x} + x_2 x e^{-2x} = e^{-2x}(c_1 + c_2 x).$$

The initial condition $y(0) = 1$ implies that $c_1 = 1$, and the initial condition $y'(0) = 4$ gives

$$y'(x) = -2e^{-2x}(1 + c_2 x) + c_2 e^{-2x} = 4 \Rightarrow c_2 = 6.$$

Hence, the unique solution to the given initial-value problem is

$$y(x) = e^{-2x}(1 + 6x).$$

8.3 The Method of Undetermined Coefficients: Annihilators

According to theorem 8.1.6, the general solution to the nonhomogeneous differential equation

$$P(D)y = F(x)$$

is of the form

$$y(x) = y_c(x) + y_p(x),$$

where y_c is the general solution to the associated homogeneous differential equation and y_p is one particular solution. We have seen in section 8.2 how y_c can be obtained, so we now turn our attention to determining a particular solution y_p. Consider the differential equation $P(D)y = F(x)$, and suppose that there is a polynomial differential operator $A(D)$ such that

$$A(D)F = 0 \Rightarrow A(D)P(D)y = 0.$$

Note that any solution to $P(D)y = F(x)$ must also solve $A(D)P(D)y = 0$. Consequently, by choosing the arbitrary constants in the general solution of the latter appropriately, we must be able to obtain a particular solution to the previous nonhomogeneous equation.

Example 8.16. We want to determine the general solution to $(D + 3)(D - 3)y = 10e^{2x}$. We first compute the auxiliary equation to obtain $y_c(x) = c_1 e^{-3x} + c_2 e^{3x}$. The nonhomogeneous term is $F(x) = 10e^{2x}$, so we need a polynomial differential operator $A(D)$ such that $A(D)F = 0$. Consider $(D - 2)e^{2x} = 0$, so we have

$$A(D) = D - 2 \Rightarrow (D - 2)(D + 3)(D - 3)y = 0,$$

which has general solution

$$y(x) = c_1 e^{-3x} + c_2 e^{3x} + A_0 e^{2x}.$$

We call $y_p(x) = A_0 e^{2x}$ a **trial solution** for the differential equation, containing the **undetermined coefficient** A_0. In order to determine the appropriate value for A_0, substitution gives

$$(D + 3)(D - 3)A_0 e^{2x} = 10e^{2x} \Rightarrow A_0(4e^{2x} - 9e^{2x}) = 10e^{2x}.$$

We must therefore choose $A_0 = -2$ to satisfy the equation. Substituting the value for A_0 yields the particular solution $y_p(x) = -2e^{2x}$, and the general solution becomes

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-3x} + c_2 e^{3x} - 2e^{2x}.$$

This technique of **method of undetrermined coefficients** is applicable only to linear differential equations that satisfy that the differential equation has constant coefficients, and that there exists a polynomial differential operator $A(D)$. Any polynomial $A(D)$ is said to **annihilate** $F(x)$, and the polynomial differential operator of lowest order is called the **annihilator** of F.
More generally, a polynomial operator $A(D)$ annihilates $F(x)$ if and only if $y = F(x)$ is a solution to

$$A(D)y = 0.$$

Thus, the only types of functions that can be annihilated by a polynomial differential operator are those that arise as solutions to a homogeneous constant coefficient linear differential equation. That is,

$$F(x) = cx^k e^{ax}, \quad F(x) = cx^k e^{ax} \sin bx, \quad F(x) = cx^k e^{ax} \cos bx.$$

$A(D) = (D-a)^{k+1}$ annihilates each of the functions $e^{ax}, xe^{ax}, \ldots, x^k e^{ax}$, so it annihilates

$$F(x) = (a_0 + a_1 x + \ldots + a_k x^k)e^{ax}$$

for all values of the constants $a_0, a_1, \ldots, a_k$; $A(D) = D^2 - 2aD + a^2 + b^2$ annihilates both of the functions $e^{ax}\cos bx$ and $e^{ax}\sin bx$, so it annihilates

$$F(x) = e^{ax}(a_0 \cos bx + b_0 \sin bx)$$

for all values a_0, b_0; $A(D) = (D^2 - 2aD + a^2 + b^2)^{k+1}$ annihilates all functions with unknowns $\cos bx$ and $\sin bx$, so it annihilates

$$F(x) = (a_0 + a_1 x + \ldots + a_k x^k)e^{ax} \cos bx + (b_0 + b_1 x + \ldots + b_k x^k)e^{ax} \sin bx.$$

Example 8.17. We want to solve the initial-value problem $y'' - y' - 2y = 10 \sin x$, $y(0) = 0$, $y'(0) = 1$. To do so, we first use the auxiliary polynomial $P(r) = (r-2)(r+1)$ to solve for $y_c(x) = c_1 e^{2x} + c_2 e^{-x}$. Considering $A(D) = D^2 - 2ad + a^2 + b^2 = D^2 + 1$, we can therefore write

$$(D^2 + 1)(D^2 - D - 2)y = 0 \Rightarrow y(x) = c_1 e^{2x} + c_2 e^{-x} + A_0 \sin x + A_1 \cos x.$$

The trial solution here is $y_p(x) = A_0 \sin x + A_1 \cos x$. Substitution into the original differential equation yields

$$(-A_0 \sin x - A_1 \cos x) - (A_0 \cos x - A_1 \sin x) - 2(A_0 \sin x + A_1 \cos x) = 10 \sin x.$$

Solving the equation gives $A_0 = -3$, $A_1 = 1$, so that $y_p(x) = -3\sin x + \cos x$. Consequently,

$$y(x) = c_1 e^{2x} + c_2 e^{-x} - 3 \sin x + \cos x.$$

Now imposing the initial conditions give the unique solution to the given initial-value problem

$$y(x) = e^{2x} - 2e^{-x} - 3 \sin x + \cos x.$$

9 Systems of Differential Equations

9.1 First-Order Linear Systems

Definition 9.1: First-order linear system

A system of differential equations of the form

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t) + b_1(t),$$

$$\vdots$$

$$\frac{\mathrm{d}x_n}{\mathrm{d}t}an1(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t) + b_n(t),$$

where $a_{ij}(t)$ and $b_i(t)$ are functions on an interval I is a **first-order linear system.** If $b_1 = b_2 = \ldots = b_n = 0$, then the system is **homogeneous**.

Consider when $n = 2$, the system reduces to

$$x_1' = a_{11}x_1 + a_{12}x_2 + b_1(t) \Rightarrow (D - a_{11})x_1 - a_{12}x_2 = b_1(t)$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + b_2(t) \Rightarrow -a_{21}x_1 + (D - a_{22})x_2 = b_2(t)$$

Example 9.2. We want to solve the system

$$x_1' = x_1 + 2x_2 \quad x_2' = 2x_1 - 2x_2.$$

To do so, rewrite the system in operator form as

$$(D-1)x_1 - 2x_2 = 0 \quad -2x_1 + (D+2)x_2 = 0$$

To eliminate x_2, operate the first equation with $D + 2$ and adding twice the second equation gives

$$(D+2)(D-1)x_1 - 4x_1 = 0 \Rightarrow (D^2 + D - 6)x_1 = 0.$$

Considering the auxiliary polynomial and solving it gives

$$x_1(t) = c_1e^{-3t} + c_2e^{2t} \quad x_2(t) = \frac{1}{2}(D-1)x_1 = \frac{1}{2}(-4c_1e^{-3t} + c_2e^{2t}.$$

Example 9.3. We now want to solve the initial-value problem

$$x_1' = x_1 + 2x_2, \quad x_2' = 2x_1 - 2x_2,$$

$$x_1(0) = 1, \quad x_2(0) = 0.$$

We solved the general case in example 9.2, so imposing the two conditions yields the following equations:

$$c_1 + c_2 = 1 \quad -4c_1 + c_2 = 0 \Rightarrow (c_1, c_2) = (0.2, 0.8).$$

Substitution gives the results: $x_1(t) = 0.2(e^{-3t} + 4e^{2t}$, $x_2(t) = 0.4(e^{2t} - e^{-3t}$.

Most systems of k differential equations that are linear in k unknown functions can be rewritten as equivalent first-order systems by redefining the independent variables.

Example 9.4. We want to rewrite the linear system

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 4y = e^t \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + t^2 \frac{\mathrm{d}x}{\mathrm{d}t} = \sin t$$

as an equivalent first-order system. To do so, consider

$$x_1 = x \quad x_2 = \frac{\mathrm{d}x}{\mathrm{d}t} \quad x_3 = y \quad x_4 = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Then, the equations can be replaced by

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} - 4x_3 = e^t \quad \frac{\mathrm{d}x_4}{\mathrm{d}t} + t^2 x_2 = \sin t$$

Considering $x_2 = \frac{\mathrm{d}x_1}{\mathrm{d}t}$ and $x_4 = \frac{\mathrm{d}x_3}{\mathrm{d}t}$ gives the following system:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_3 + e^t \quad \frac{\mathrm{d}x_3}{\mathrm{d}t} = x_4 \quad \frac{\mathrm{d}x_4}{\mathrm{d}t} = -t^2 x_2 + \sin t.$$

9.2 Vector formulation

Consider the system of equations

$$x_1' = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t)$$

$$\vdots$$

$$x_n' = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t).$$

This system can be written as the equivalent vector equation

$$x'(t) = A(t)x(t) + b(t),$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix} \quad b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

Let $V_n(I)$ denote the set of all n-vector functions defined on an interval I. Note that $V_n(I)$ is a vector space.

Definition 9.5: Wronskian

Let $x_1(t), x_2(t), \ldots, x_n(t)$ be vectors in $V_n(I)$. The **Wronskian** of these vector functions is defined by

$$W\left[x_1, x_2, \ldots, x_n\right](t) = \det\left(\begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \end{bmatrix}\right)$$

Theorem 9.6

Let $x_1(t), x_2(t), \ldots, x_n(t)$ be vectors in $V_n(I)$. If $W[x_1, x_2, \ldots, x_n](t_0)$ is nonzero at some $t_0 \in I$, then $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ is linearly independent on I.

Proof. Consider $c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t) = 0$. This is equivalent to the vector equation

$$X(t)c = 0,$$

where $c = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}^T$ and $X(t) = \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \end{bmatrix}$. Assuming

$$\det([X(t_0)]) = W[x_1, x_2, \ldots, x_n](t_0) \neq 0,$$

the only solution to this $n \times n$ system of linear equations is $c = 0$. Consequently, $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ is linearly independent on I if such t_0 exists in I. □

Definition 9.7: Vector differential equation

A system of linear differential equations written in the vector form

$$x'(t) = A(t)x(t) + b(t)$$

is a **vector differential equation**.

The solutions to the general first-order linear system of differential equations is equivalent to solving for all column vector functions $x(t) \in V_n(I)$ that satisfies

$$x'(t) = A(t)x(t) + b(t)$$

9.3 General Results for First-Order Linear Differential Systems

Theorem 9.8

The initial value problem

$$x'(t) = A(t)x(t) + b(t), \quad x(t_0) = x_0,$$

where $A(t)$ and $b(t)$ are continuous on an interval I, has a unique solution on I.

Proof. The proof is omitted. □

Consider the homogeneous vector differential equations $x'(t) = A(t)x(t)$ first. Here, A is an $n \times n$ matrix function.

Theorem 9.9

The set of all solutions to $x'(t) = A(t)x(t)$, where $A(t)$ is an $n \times n$ matrix function that is continuous on an interval I, is a vector space of dimension n.

Proof. Consider the set of all solutions S satisfying $x' = A(t)x(t)$. Considering $T(x) = Ax$ and $D(x) = x'$ as linear transformations, so is

$$(D - T)(x) = x' - Ax.$$

Since $S = \ker(D - T)$, it is definitely a subspace of $V_n(I)$.
Now proving $\dim[S] = n$ by constructing a basis for S. Claim that there exist n linearly independent solutions to $x' = Ax$. Let e_i denote the i-th column vector of the identity matrix I_n, then the initial value problem

$$x_i'(t) = A(t)x_i(t) \quad x_i(t_0) = e_i \quad i = 1, 2, \ldots, n$$

has a unique solution $x_i(t)$. Furthermore, $W[x_1, x_2, \ldots, x_n](t_0) = \det(I_n) = 1 \neq 0$ for any $t_0 \in I$, so that $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ is linearly independent on I. Now let $x(t)$ be any real solution to $x' = Ax$ on I. Since $x_1(t_0), x_2(t_0), \ldots, x_n(t_0)$ is the standard basis for $\mathbb{R}^n$, there is

$$x(t_0) = c_1x_1(t_0) + c_2x_2(t_0) + \ldots + c_nx_n(t_0)$$

for some scalars $c_1, c_2, \ldots, c_n$. Therefore, $x(t)$ is the unique solution to the initial value problem

$$x'(t) = A(t)x(t) \quad x(t_0) = c_1x_1(t_0) + c_2x_2(t_0) + \ldots + c_nx_n(t_0).$$

Considering $x(t) = u(t) = c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t)$. Any solution to $x' = Ax$ on I can be written as a linear combination of the n linearly independent solutions $x_1(t), x_2(t), \ldots, x_n(t)$, and hence, $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ is a basis for the solution space. The proof follows. □

Definition 9.10: Fundamental solution set

Let $A(t)$ be an $n \times n$ matrix function that is continuous on an interval I. Any set of n solutions, $\{x_1, x_2, \ldots, x_n\}$, to $x' = Ax$ that is linearly independent on I is a **fundamental solution set** on I. The corresponding matrix $X(t)$ defined by

$$X(t) = \begin{bmatrix} x_1 & x_2 & \ldots x_n \end{bmatrix}$$

is a **fundamental matrix** for the vector differential equation $x' = Ax$.

Example 9.11. Consider the vector differential equation

$$x' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} x,$$

and let

$$x_1(t) = \begin{bmatrix} -e^t \cos 2t \\ e^t \sin 2t \end{bmatrix} \quad x_2(t) = \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}.$$

We want to verify that $\{x_1, x_2\}$ is a fundamental set of solutions, and calculate both the general solution and the initial value problem $x' = Ax$, $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. To verify the fundamental set of solutions, consider

$$x_1' = \begin{bmatrix} e^t(-\cos 2t + 2\sin 2t) \\ e^t(\sin 2t + 2\cos 2t) \end{bmatrix} \quad x_2' = \begin{bmatrix} e^t(\sin 2t + 2\cos 2t) \\ e^t(\cos 2t - 2\sin 2t) \end{bmatrix}.$$

It follows that $x_1' = Ax_1$ and $x_2' = Ax_2$. Furthermore,

$$W[x_1, x_2](t) = \begin{vmatrix} -e^t \cos 2t & e^t \sin 2t \\ e^t \sin 2t & e^t \cos 2t \end{vmatrix} = -e^{2t} \neq 0.$$

Hence, $\{x_1, x_2\}$ is linearly independent on any interval. Hence, it is a fundamental set of solutions for the given vector differential equation. The general solution is then

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = \begin{bmatrix} e^t(-c_1 \cos 2t + c_2 \sin 2t) \\ e^t(c_1 \sin 2t + c_2 \cos 2t) \end{bmatrix}.$$

Imposing the initial condition gives

$$\begin{bmatrix} -c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

so that $c_1 = -3$ and $c_2 = 2$. Hence,

$$x_1(t) = e^t(3\cos 2t + 2\sin 2t) \quad x_2(t) = e^t(-3\sin 2t + 2\cos 2t).$$

Now consider nonhomogeneous vector differential equations.

Theorem 9.12

Let $A(t)$ be a matrix function that is continuous on an interval I, and let $\{x_1, x_2, \ldots, x_n\}$ be a fundamental solution set on I for the vector differential equation $x'(t) = A(t)x(t)$. If $x_p(t)$ is any particular solution to the nonhomogeneous vector differential equation

$$x'(t) = A(t)x(t) + b(t)$$

on I, then every solution on I is of the form

$$x(t) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + x_p.$$

Proof. Since $x(p)$ is a solution,

$$x_p'(t) = A(t)x_p(t) + b(t).$$

Now consider any other solution $u(t)$ such that $u'(t) = A(t)u(t) + b(t)$. Subtracting the two gives

$$(u - x_p)' = A(u - x_p).$$

Thus, the vector function $x = u - x_p$ is a solution to the associated homogeneous system $x' = Ax$ on I / Since $\{x_1, x_2, \ldots, x_n\}$ spans the solution space, it follows that

$$u - x_o = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \Rightarrow u = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + x_p.$$

□

9.4 Vector Differential Equations: Nondefective Coefficient Matrix

Consider homogeneous linear systems $x' = Ax$ where A is an $n \times n$ matrix of real constants. Consider example 9.2 in vector form, where the solution is

$$x_1(t) = \begin{bmatrix} e^{-3t} \\ -2e^{-3t} \end{bmatrix} \quad x_2(t) = \begin{bmatrix} e^{2t} \\ \frac{1}{2}e^{2t}. \end{bmatrix}$$

Note that both of the solutions are of the form $x(t) = e^{\lambda t}v$, where λ is a scalar and v is a constant vector. Now, differentiating $x(t) = e^{\lambda t}v$ with respect to t yields

$$x' = \lambda e^{\lambda t}v = Ax.$$

Thus, $x(t) = e^{\lambda t}v$ is a solution if and only if

$$\lambda e^{\lambda t}v = e^{\lambda t}Av \Rightarrow Av = \lambda v.$$

Theorem 9.13

Let A be an $n \times n$ matrix of real constants, and let λ be an eigenvalue of A with corresponding eigenvector v. Then,

$$x(t) = e^{\lambda t}v$$

is a solution to the constant coefficient vector differential equation $x' = Ax$ on any interval.

Example 9.14. Consider the general solution to

$$x_1' = 2x_1 + x_2 \quad x_2' = -3x_1 - 2x_2.$$

The corresponding vector differential equation is

$$x' = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} x \Rightarrow \det(A - \lambda I) = \lambda^2 - 1.$$

It follows that A has eigenvalues $\lambda = \pm 1$.
Eigenvalue $\lambda_1 = 1$: $(A - \lambda_1 v) = 0$ has solution $v = r(1, -1)$. Therefore,

$$x_1(t) = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Eigenvalue $\lambda_2 = -1$: $(A - \lambda_2 v) = 0$ has solution $v = s(1, -3)$. Therefore,

$$x_2(t) = e^{-t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Considering the Wronskian, $W[x_1, x_2](t) = -2 \neq 0$, so that $\{x_1, x_2\}$ is linearly independent on any interval. The general solution is therefore

$$x(t) = \begin{bmatrix} c_1e^t + c_2e^{-t} \\ -c_1e^t - 3c_2e^{-t} \end{bmatrix}$$

Theorem 9.15

Let A be an $n \times n$ matrix of real constants. If A has n real linearly independent eigenvectors $v_1, v_2, \dots, v_n$ with corresponding real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, not necessarily distinct, then the vector functions $\{x_1, x_2, \dots, x_n\}$ defined by

$$x_k(t) = e^{\lambda_k t} v_k, \quad k = 1, 2, \dots, n,$$

for all t, are linearly independent solutions to $x' = Ax$ on any interval. The general solution to this vector differential equation is

$$x(t) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n.$$

Proof. Each $x_k(t)$ satisfies $x' = Ax$. Now consider

$$W[x_1, x_2, \dots, x_n] = e^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)t} \det\left(\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}\right) \neq 0.$$

Since the eigenvectors are linearly independent by assumption, the solutions are linealry independent on any interval. Thus, $\{x_1, x_2, \dots, x_n\}$ is a fundamental solution set to the vector differential equation. □

Generally, let A be an $n \times n$ matrix of real constants. Suppose λ is a real eigenvalue of A with corresponding linearly independent eigenvectors $v_1, v_2, \dots, v_k$, then k linearly independent solutions to $x' = Ax$ are $x_j(t) = e^{\lambda t} v_j$, $j = 1, 2, \dots, k$. Suppose $\lambda = a + ib$ is a complex eigenvalue of A with corresponding linearly independent eigenvectors $v_1, v_2, \dots, v_k$, where $v_j = r_j + is_j$, then $2k$ real-valued linearly independent solutions to $x' = Ax$ are

$$x_{11}(t) = e^{at}(\cos bt r_1 - \sin bt s_1) \quad x_{12}(t) = e^{at}(\sin bt r_1 + \cos bt s_1)$$

$$\vdots$$

$$x_{k1}(t) = e^{at}(\cos bt r_k - \sin bt s_k) \quad x_{k2}(t) = e^{at}(\sin bt r_k + \cos bt s_k).$$