Boyd Convex Optimization Exercises

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1 Chapter 2

Problem 1: Boyd 2.2

Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

Solution. The intersection of two convex sets is convex. Therefore if S is a convex set, the intersection of S with a line is also convex. Conversely, suppose the intersection of S with any line is convex. Then for any two distinct points $x_1, x_2 \in S$, the intersection of S with the line through x_1, x_2 is convex. The convex combination is contained in the intersection, hence in S.

Problem 2: Boyd 2.4

Show that the convex hull of a set S is the intersection of all convex sets that contain S. (The same method can be used to show that the conic, or affine, or linear hull of a set S is the intersection of all conic sets, or affine sets, or subspaces that contain S.)

Solution. Let H be the convex hull of S and A be the intersection of all convex sets that contain S. It suffices to prove that $H \subseteq A$ and $A \subseteq H$.
First we show $H \subseteq A$. Suppose $x \in H$. Now let D be any convex set such that $S \subseteq D$, so all points in S are in D. Since D is convex, and x is a convex combination of $x_1, x_2, ..., x_n$, $x \in D$. It follows that $x \in A$. Conversely, since H is convex and contains S, then we must have $H = D$ for some D.

Problem 3: Boyd 2.5

What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n : a^T x = b_1\}$ and $\{x \in \mathbb{R}^n : a^T x = b_2\}$?

Solution. Take two points $x_1 = (\frac{b_1}{\|a\|_2^2})a$, $x_2 = (\frac{b_2}{\|a\|_2^2})a$. The distance is $d = \|x_1 - x_2\|_2 = |b_1 - b_2|/\|a\|_2$.

Problem 4: Boyd 2.11

Hyperbolic sets. Show that the hyperbolic set $\{x \in \mathbb{R}_+^2 : x_1 x_2 \geq 1\}$ is convex. As a generalization, show that $\{x \in \mathbb{R}_+^n : \prod_{i=1}^n x_i \geq 1\}$ is convex. *Hint.* If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b$.

Solution. Consider a convex combination z of two points (x_1, x_2) and : (y_1, y_2) in the set. If $x \geq y$, then $z = \theta x + (1-\theta)y \geq y$. Suppose $y \ngeq 0$ and $x \ngeq y$ Then

$$\begin{aligned}(\theta x_1 + (1-\theta y_1)(\theta x_2 + (1-\theta)y_2) &= \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta)x_1 y_2 + \theta(1-\theta)x_2 y_1 \\ &= \theta x_1 x_2 + (1-\theta)y_1 y_2 - \theta(1-\theta)(y_1 - x_1)(y_2 - x_2) \\ &\geq 1.\end{aligned}$$

(b) Assume that $\prod_i x_1 \geq 1$ and $\prod_i y_1 \geq 1$. Then

$$\prod_i (\theta x_1 + (1-\theta)y_i) \geq \prod x_i^\theta y_i^(1-\theta) = (\prod_i x_i)^\theta (\prod_i y_i)^{1-\theta} \geq 1.$$

Problem 5: Boyd 2.16

Show that if S_1 and S_2 are convex sets in $\mathbb{R}^{m \times n}$, then so is their partial sum
$S = \{(x, y_1 + y_2) : x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}$.

Solution. Consider two points $(x_1, y_{11} + y_{12})$, $(x_2, y_{21} + y_{22}) \in S$. For $0 \leq \theta \leq 1$,
$\theta(x_1, y_{11} + y_{12}) + (1-\theta)(x_2, y_{21} + y_{22}) = (\theta x_1 + (1-\theta)x_2, (\theta y_{11} + (1-\theta)y_{21} + (\theta y_{12} + (1-\theta)y_{22}))$ is in S by the convexity of S_1 and S_2.

Problem 6: Boyd 2.23

Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

Solution. $C = \{x \in \mathbb{R}^2 : x_2 \leq 0\}$ and $D = \{x \in \mathbb{R}_+^2 : x_1 x_2 \geq 1\}$.

Problem 7: Boyd 2.29

Cones in $\mathbb{R}^2$. Suppose $K \in \mathbb{R}^2$ is a closed convex cone.
(a) Give a simple description of K in terms of the polar coordinates of its elements ($x = r(\cos\phi, \sin\phi)$ with $r \geq 0$.
(b) Give a simple description of K^*, and draw a plot illustrating the relation between K and K^*.
(c) When is K pointed?
(d) When is K proper (hence, defines a generalized inequality)? Draw a plot illustrating what $x \preceq_K y$ means when K is proper.

Solution. (a) In $\mathbb{R}^2$, a cone K is a fraction of $\mathbb{R}^2$. It can be expressed as $K = \{(r\cos\phi, r\sin\phi : r \geq 0, \alpha \leq \phi \leq \beta\}$. If $\beta - \alpha = 180°$, this gives a halfspace.
(b) By definition, K^* is the intersection of all halfspaces $x^T y \geq 0$ where $x \in K$. However, if K is pointed, then $K^* = \{y : y_1 \cos\alpha + y_2 \sin\alpha \geq 0, y_1 \cos\beta + y_2 \sin\beta \geq 0\}$. If K is a halfspace, $K = \left\{x : v^T x \geq 0\right\}$, $K^* = \{tv : t \geq 0\}$.
(c) K is pointed when $\beta - \alpha = 180°$.
(d) In order for K to be proper, it has to be closed, convex, and pointed. The interior of K should also be nonempty. $K = \{r\cos\phi, r\sin\phi : r \geq 0, \alpha \leq \phi \leq \beta\}$

Problem 8: Boyd 2.32

Find the dual cone of $\{Ax : x \geq 0\}$, where $A \in \mathbb{R}^{m\times n}$.

Solution. $K^* = \left\{y : A^T y \geq 0\right\}$.

Problem 9: Boyd 2.36

Euclidean distance matrices. Let $x_1, \ldots, x_n \in \mathbb{R}^k$. The matrix $D \in \mathbb{S}^n$ defined by $D_{ij} = \|x_i - x_j\|_2^2$ is called a Euclidean distance matrix. It satisfies some obvious properties such as $D_{ij} = D_{ji}$, $D_{ii} = 0$, $D_{ij} \geq 0$, and $D_{ik}^{\frac{1}{2}} \leq D_{ij}^{\frac{1}{2}} + D_{jk}^{\frac{1}{2}}$. We now pose the question: When is a matrix $D \in \mathbb{S}^n$ a Euclidean distance matrix (for some points in $\mathbb{R}^k$, for some k? A famous result answers this question: $D \in \mathbb{S}^n$ is a Euclidean distance matrix if and only if $D_{ii} = 0$ and $x^Y Dx \leq 0$ for all x with $1^T x = 0$.
Show that the set of Euclidean distance matrices is a convex cone. Find the dual cone.

Solution. The set of Euclidean distance matrices in $\mathbb{S}^n$ is a closed convex cone because it is the intersection of infinitely many halfspaces defined by $e_i^T De_i \leq 0$, $e_i^T De_i \geq 0$, $x^T Dx = \sum_{j,k} x_j x_j D_{jk} \leq_0$, for all $i = 1, \ldots, n$, and all x with $1^T x = 1$.
The dual cone is $K^* = \mathrm{conv}(\left\{-\times^T : 1^T x = 1\right\} \cup \left\{e_1 e_1^T, -e_1 e_1^T, \ldots, e_n e_n^T, -e_n e_n^T\right\})$.

2 Chapter 3

Problem 10: Boyd 3.2

Level sets of convex, concave, quasiconvex, and quasiconcave functions. Some level sets of a function f are shown below. Could f be convex, concave, quasiconvex, quasiconcave? Explain your answer.

Solution. For the first graph, the level sets are convex, so the function might be convex or quasiconvex. For the second graph, the level sets are convex, so the function might be concave or quasiconcave.

Problem 11: Boyd 3.3

Inverse of an increasing convex function. Suppose $F : \mathbb{R} \to \mathbb{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse, i.e., the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $x \in (a, b)$. What can you say about convexity or concavity of g?

Solution. The hypograph of g is **hypo** $g = \{(y,t) : t \leqslant g(y)\} = \{(y,t) : f(t) \leqslant y\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **epi** f, so g is concave.

Problem 12: Boyd 3.16

For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

(c) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$.

(d) $f(x_1, x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$.

(e) $f(x_1, x_2) = \frac{x_1^2}{x_2}$ on $\mathbb{R} \times \mathbb{R}_{++}$.

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $\alpha \in [0, 1]$, on $\mathbb{R}^2_{++}$.

Solution. (a) Convex, quasiconvex, quasiconcave.

(b) $\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is neither positive semidefinite nor negative semidefinite. Hence, f is neither convex nor concave. It is quasiconcave because superlevel sets $\left\{(x_1, x_2) \in \mathbb{R}^2_{++} : x_1 x_2 \geqslant \alpha\right\}$ are convex.

(c) $\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2x_1^{-2} & x_1^{-1} x_2^{-1} \\ x_1^{-1} x_2^{-1} & 2x_2^{-2} \end{bmatrix} \succeq 0$, so $f(x)$ is convex and quasiconvex.

(d) $\nabla^2 f(x) = \begin{bmatrix} 0 & -x_2^{-2} \\ x_2^{-2} & 2x_1 x_2^{-3} \end{bmatrix}$ is neither positive semidefinite nor negative semidefinite. Therefore, f is neither convex nor concave. It is quasiconvex and quasiconcave, so it's quasilinear.

(e) $\nabla^2 f(x) = \begin{bmatrix} 2x_2^{-1} & -2x_1 x_2^{-2} \\ -2x_1 x_2^{-2} & 2x_1^2 x_2^{-3} \end{bmatrix} \succeq 0$, so f is convex and quasiconvex.

(f) $\nabla^2 f(x) = \begin{bmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & (1-\alpha)(-\alpha) x_1^\alpha x_2^{-\alpha-1} \end{bmatrix} = \alpha(1-\alpha) x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} -x_1^{-2} & x_1^{-1} x_2^{-1} \\ x_1^{-1} x_2^{-1} & x_2^{-2} \end{bmatrix} \preceq 0$. Therefore, f is concave and quasiconcave.

Problem 13: Boyd 3.21

Pointwise maximum and supremum. Show that the following functions $f : \mathbb{R}^n \to R$ are convex.

(a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)} x - b^{(i)}\|$, where $A^{(i)} \in \mathbb{R}^{m \times n}$, $b^{(i)} \in \mathbb{R}^m$ and $\|\cdot\|$ is a norm on $\mathbb{R}^m$.

(b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbb{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$, and $|x|_{[i]}$ is the i th largest component of $|x|$. In other words, $|x|_{[1]}$, $|x|_{[2]}$, ..., $|x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

Solution. (a) f is the maximum of k functions $\|A^{(i)} x - b^{(i)}\|$. Each of those functions is convex because it is a composition of an affine transformation and a norm. f is therefore convex.

(b) $f(x)$ is the maximum of $\binom{n}{r}$ convex functions, so it is convex.

Problem 14: Boyd 3.29

Representation of piecewise-linear convex functions. A function $f : \mathbb{R}^n \to \mathbb{R}$, with $\mathbf{dom} f = \mathbb{R}^n$, is called piecewise-linear if there exists a partition of $\mathbb{R}^n$ as $\mathbb{R}^n = X_1 \cup X_2 \cup \ldots \cup X_L$, where $\mathbf{int} X_i \neq \varnothing$ and $\mathbf{int} X_i \cap \mathbf{int} X_j = \varnothing$ for $i \neq j$ and a family of affine functions $a_1^T x + b_1, \ldots, a_L^T x + b_L$ such that $f(x) = a_i^T x + b_1, \ldots, a_L^T x + b_L$. Show that this means that $f(x) = \max\left\{a_1^T x + b_1, \ldots, a_L^T x + b_L\right\}$.

Solution. By Jensen's inequality, for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$,

$$f(y + t(x-y)) \leqslant f(y) + t(f(x) - f(y)) \Rightarrow f(x) \geqslant f(y) + [f(y + t(x-y)) - f(y)]/t.$$

Suppose $x \in X_i$, choose any $y \in \mathbf{int} X_j$, for some j, so

$$a_i^T x + b_i \geqslant a_j^T y + b_j + [a_j^T (y + t(x-y)) + b_j - a_j^T y - b_j]/t = a_j^T x + b_j.$$

The above inequality is true for any j, so $a_i^T x + b_i \geqslant \max(a_j^T x + b_j)$, and the equality holds when the maximum of right hand side is taken.

Problem 15: Boyd 3.30

Convex hull or envelope of a function. The convex hull or convex envelope of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$g(x) = \inf\{t : (x,t) \in \mathbf{conv\ epi} f\}.$$

Geometrically, the epigraph of g is the convex hull of the epigraph of f. Show that g is the largest convex underestimator of f. In otherwords, show that if h is convex and satisfies $h(x) \leqslant f(x)$ for all x, then $h(x) \leqslant g(x)$ for all x.

Solution. g is convex by the epigraph. Let h be a convex lower bound on f. Since h is convex, its epigraph is a convex set, and the epigraph of f is a subset of epigraph of h since h is a lower bound on f. The convex hull of a set is the intersection of all the convex sets that contain the set, so $\mathbf{conv\ epi} f = \mathbf{epi} g \subset \mathbf{epi} h$.

Problem 16: Boyd 3.32

Products and ratios of convex functions. In general the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbb{R}$. Prove the following.

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

(b) If f and g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.

(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

Solution. (a) For $0 \leqslant \theta \leqslant 1$,

$$f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) \leqslant (\theta f(x) + (1-\theta)f(y))(\theta g(x) + (1-\theta g(y))$$
$$= \theta f(x)g(x) + (1-\theta)f(y)g(y) + \theta(1-\theta)(f(y)-f(x))(g(x)-g(y)$$

The third term is less than or equal to zero if f and g are both increasing or both decreasing. Therefore,

$$f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) \leqslant \theta f(x)g(x) + (1-\theta)f(y)g(y).$$

(b) Same as (a), but reverse the inequalities.

(c) $1/g$ is convex, positive and increasing, so same as (a).

Problem 17: Boyd 3.34

The Minkowski function. The Minkowski function of a convex set C is defined as

$$M_C(x) = \inf\left\{t > 0 : t^{-1}x \in C\right\}.$$

(a) Draw a picture giving a geometric interpretation of how to find $M_C(x)$.

(b) Show that M_C is homogeneous, i.e., $M_C(\alpha x) = \alpha M_C(x)$ for $\alpha \geqslant 0$.

(c) What is **dom** M_C?

(d) Show that M_C is a convex function.

(e) Suppose C is also closed, symmetric (if $x \in C$ then $-x \in C$), and has nonempty interior. Show that M_C is a norm. What is the corresponding unit ball?

Solution. (a)

(b) If $\alpha > 0$, then $M_C(\alpha x) = \inf\left\{t > 0 : t^{-1}\alpha x \in C\right\} = \alpha \inf\left\{t/\alpha > 0 : t^{-1}\alpha x \in C\right\} = \alpha M_C(x)$.

If $\alpha = 0$, then $M_C(\alpha x) = 0$ if $0 \in C$, ∞ if $0 \notin C$.

(c) **dom** $M_C = \{x : x/t \in C\}$ for some $t > 0$.

(d) **dom** M_C is a convex set. Suppose $x, y \in$ **dom** M_C. The convexity of C tells that

$$\frac{\theta x + (1-\theta)y}{\theta t_x + (1-\theta)t_y} = \frac{\theta t_x(x/t_x) + (1-\theta)t_y(y/t_y)}{\theta t_x + (1-\theta)t_y} \in C.$$

This is true for any $t_x, t_y > 0$. Then,

$$M_C(\theta x + (1-\theta)y) \leqslant \theta M_C(x) + (1-\theta)M_C(y).$$

(e) It the norm with unit ball C.

Problem 18: Boyd 3.35

Support function calculus. Recall that the support function of a set $C \subseteq \mathbb{R}^n$ is defined as $S_C(y) = \sup\left\{y^T x : x \in C\right\}$. On page 81 we showed that S_C is a convex function.

(a) Show that $S_B = S_{\text{conv}B}$.

(b) Show that $S_{A+B} = S_A + S_B$.

(c) Show that $S_{A\cup B} = \max\{S_A, S_B\}$.

(d) Let B be closed and convex. Show that $A \subseteq B$ if and only if $S_A(y) \leqslant S_B(y)$ for all y.

Solution. (a) Let $A = \textbf{conv}\, B$. Since $B \subseteq A$, $S_B(y) \leqslant S_A(y)$. Suppose we have strict inequality $y^T u < y^T v$ for all $u \in B$ and some $v \in A$, a contradiction. It follows that $S_B(y) = S_A(y)$.
(b) $S_{A+B}(y) = \sup\left\{y^T(u+v) : u \in A, v \in B\right\} = \sup\left\{y^T u : u \in A\right\} + \sup\left\{y^T v : u \in B\right\} = S_A(y) + S_B(y)$.
(c) $S_{A\cup B}(y) = \sup\left\{y^T u : u \in A \cup B\right\} = \max\{S_A(y), S_B(y)\}$.
(d) Suppose $A \nsubseteq B$, then exists $x_0 \in A$ and $x_0 \notin B$. Since B is closed and convex, x_0 can be separated by a hyperplane. It follows that $S_B(y) < y^T x_0 \leqslant S_A(y)$.

3 Chapter 4

Problem 19: Boyd 4.2

Consider the optimization problem

$$\min f_0(x) = -\sum_{i=1}^m \log(b_i - a_i^T x)$$

with domain $\textbf{dom} f_0 = \{x : Ax \prec b\}$, where $A \in \mathbb{R}^{m\times n}$. We assume that the domain is nonempty. Prove the following facts (which include the results quoted without proof on page 141).

(a) $\textbf{dom} f_0$ is unbounded if and only if there exists a $v \neq 0$ with $Av \preceq 0$.

(b) f_0 is unbounded below if and only if there exists a v with $Av \preceq 0$, $Av \neq 0$ and only if there exists no $z \succ 0$ such that $A^T z = 0$. This follows from the theorem of alternatives in example 2.21, page 50.

(c) If f_0 is bounded below then its minimum is attained, i.e., there exists an x that satisfies the optimality condition in (4.23).

(d) The optimal set is affine: $X_{opt} = \{x^* + v : Av = 0\}$, where x^* is any optimal point.

Solution. Because the domain is nonempty, we assume $x_0 \in \textbf{dom} f$.

(a) If such a v exists, then $\textbf{dom} f_0$ is unbounded, since $x_0 + tv \in \textbf{dom} f_0$ for all $t \geqslant 0$. Conversely, suppose x^k is a sequence of points in $\textbf{dom} f_0$ with norm approaching infinity. Define $v^k = x^k/\|x^k\|_2$. The sequence has a convergent subsequence because $\|v^k\|_2 = 1$ for all k. Consider its limit v gives $\|v\|_2 = 1$. Since for all k,

$$a_i^T v^k < b_i/\|x_k\|_2,$$

there is $a_i^T v \leqslant 0$, therefore $Av \preceq 0$ and $v \neq 0$.

(b) If there exists such a v, let j be such that $a_j^T v < 0$. For $t \geqslant 0$,

$$f_0(x_0 + tv) = -\sum_{i=1}^{m} \log(b_i - a_i^T x_0 - ta_i^T v) \leqslant -\sum_{i \neq j}^{m} \log(b_i - a_i^T x_0) - \log(b_j - a_j^T x_0 - ta_j^T v).$$

Clearly, the right hand side decreases without bound as t increases, hence f_0 is not bounded below.

Conversely, suppose f is unbounded below. Let x^k be a sequence with $b - Ax^k > 0$, and $f_0(x^k) \to -\infty$. By the convexity,

$$f_0(x^k) \geqslant f_0(x_0) + \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x_0} a_i^T (x^k - x_0) = f_0(x_0) + m - \sum_{i=1}^{m} \frac{b_i - a_i^T x^k}{b_i - a_i^T x_0},$$

so if $f_0(x^k) \to -\infty$, $\max_i (b_i - a_i^T x^k) \to -\infty$.

(c) Assume that $\text{rank}(A) = n$. If $\textbf{dom} f_0$ is bounded, then the result follows from the fact that the sublevel sets of f_0 are closed. If $\textbf{dom} f_0$ is unbounded, let v be a direction in which it is unbounded, $Av \preceq 0$. Since $\text{rank}(A) = 0$, $Av \neq 0$, implying f_0 is unbounded. Then, if $\text{rank}(A) = n$, then f_0 is unbounded below if and only if its domain is bounded, and therefore its minimum is attained.

(d) Consider $\text{rank}(A) = n$. The hessian matrix of f_0 at x is

$$\nabla^2 f(x) = A^T \text{diag}(d) A \quad d_i = \frac{1}{(b_i - a_i^T)^2},$$

which is positive definite if $\text{rank}(A) = n$, i.e., f_0 is strictly convex. Therefore, the optimal point is unique.

Problem 20: Boyd 4.6

Handling convex equality constraints. A convex optimization problem can have only linear equality constraint functions. In some special cases, however, it is possible to handle convex equality constraint functions, i.e., constraints of the form $g(x) = 0$, where g is convex. We explore this idea in this problem.
Consider the optimization problem of minimizing $f_0(x)$ subject to

$$f_i(x) \leqslant 0 \quad h(x) = 0,$$

where f_i and h are convex functions with domain $\mathbb{R}^n$. Unless h is affine, this is not a convex optimization problem. Consider the related problem of minimizing $f_0(x)$ subject to

$$f_i(x) \leqslant 0 \quad h(x) \leqslant 0,$$

where the convex equality constraint has been relaxed to a convex inequality. This problem is, of course, convex.
Now suppose we can guarantee that at any optimal solution x^* of the convex problem, we have $h(x^*) = 0$, i.e., the inequality $h(x) \geqslant 0$ is always active at the solution. Then we can solve the (nonconvex) problem of the equality by solving the convex problem of the inequality. Show that this is the case if there is an index r such that f_0 is monotonically increasing in x_r, $f_1, \ldots, f_m$ are nonincreasing in x_r, and h is monotonically decreasing in x_r.

Solution. Suppose x^* is optimal, and $h(x^*) < 0$. We can then decrease x_r while staying in the boundary of g. By decreasing x_r, we decrease the objective, preserve the inequalities $f_i(x) \leqslant 0$, and increase the function h.

Problem 21: Boyd 4.8

Some simple LPs. Give an explicit solution of each of the following LPs.

(a) *Minimizing a linear function* $c^T x$ *over an affine set* $Ax = b$.

(b) *Minimizing a linear function* $c^T x$ *over a halfspace* $a^T x \leqslant b$.

(c) *Minimizing a linear function* $c^T x$ *over a rectangle* $l \preceq x \preceq u$.

(d) *Minimizing a linear function* $c^T x$ *over a probability simplex* $1^T x = 1, x \succeq 0$.

(e) *Minimizing a linear function* $c^T x$ *over a unit box with a total budget constraint* $1^T x \leqslant \alpha, 0 \preceq x \preceq 1$.

(f) *Minimizing a linear function* $c^T x$ *over a unit box with a weighted budget constraint* $d^T x = \alpha$.

Solution. Do this problem part by part.

(a) Consider the normal case, where c is orthogonal to the nullspace of A. Then $c = A^T\lambda + \hat{c}$, $A\hat{c} = 0$. Then if $\hat{c} = 0$, on the feasible set the objective function reduces to

$$c^T x = \lambda^T Ax + \hat{c}^T x = \lambda^T b.$$

The optimal value is $\lambda^T b$. Otherwise, if $b \notin \mathcal{R}(A)$, the problem is infeasible and the optimal value is ∞, orelse the optimal value is unbounded negatively so the optimal value is $-\infty$.

(b) This is always feasible, so $c = a\lambda + \hat{c}$, where $a^T\hat{c} = 0$. IF $\lambda > 0$, the problem is bounded below. Choose $x = -ta$, and $c^T x \to -\infty$ and $a^T x - b = -ta^T a - b \leqslant 0$ for large t. If $\hat{c} \neq 0$, the problem is unbounded below. If $c = a\lambda$ for some $\lambda \leqslant 0$, the optimal value is $c^T ab = \lambda b$.

(c) The problem can be solved by separating components, minimizing over each component of x independently. The optimal x_i^* minimizes $c_i x_i$ subject to the constraint $l_i \leqslant x_i \leqslant u_i$. If $c_i > 0$, when $x_i^* = l_i$. If $c_i < 0$, then $x_i^* = u_i$. If $c_i = 0$, then any x_i in the interval $[l_i, u_i]$ is optimal. Therefore,

$$p^* = l^T c^+ + u^T c^-,$$

where $c_i^+ = \max\{c_i, 0\}$, $c_i^- = \max\{-c_i, 0\}$.

(d) Suppose the components of c re sorted in increasing order, then $c^T x \geqslant c_1(1^T x) = c_{\min}$ for all feasible x. The optimal value is then $p^* = c_{\min}$.

(e) In the case of an inequality constraint $1^T x \leqslant \alpha$ with α being an integer, the optimal value is the sum of the α smallest nonpositive coefficients of c. If α is not an integer, the sum is taken to $c_{\lfloor\alpha\rfloor}$ and a term $c_{1+\lfloor\alpha\rfloor}(a - \lfloor\alpha\rfloor)$ is added.

(f) Considering $y_i = d_i x_i$, the problem becomes minimizing $\sum_{i=1}^n (c_i/d_i) y_i$ subject to $1^T x = \alpha$. Sort the ratios in increasing order gives

$$\frac{c_1}{d_1} \leqslant \frac{c_2}{d_2} \leqslant \ldots \leqslant c_{\frac{n}{d_n}},$$

To minimize, consider $y_1 = d_1, \dots, y_k = d_k, y_{k+1} = \alpha - (d_1 + \dots + d_k), y_{k+2} = \dots = y_n = 0$, where $k = \max\{i \in \{1, \dots, n\} : d_1 + \dots + d_i \leqslant \alpha\}$.

Problem 22: Boyd 4.9

Square LP. Consider the LP minimize $c^T x$ subject to $Ax \preceq b$ with A square and nonsingular. Show that the optimal value is given by

$$p^* = c^T A^{-1} b$$

for $A^{-T} c \preceq 0$, and $-\infty$ otherwise.

Solution. Consider $y = Ax$. Then the problem is equivalent to minimizing $c^T A^{-1} y$ subject to $y \preceq b$. If $A^{-T} c \preceq 0$, the optimal solution is $y = b$, with $p^* = c^T A^{-1} b$. Otherwise, the LP is unbounded below.

Problem 23: Boyd 4.21

Some simple QCQPs. Give an explicit solution of each of the following QCQPs.

(a) *Minimizing a linear function $c^T x$ over an ellipsoid $x^T A x \leqslant 1$ centered at the origin. What is the solution is not convex ($A \notin \mathbb{S}^n_+$)?*

(b) *Minimizing a linear function $c^T x$ over an ellipsoid $(x - x_c)^T A (x - x_c) \leqslant 1$, where $A \in \mathbb{S}^n_{++}$ and $c \neq 0$.*

(c) *Minimizing a quadratic form $x^T B x$ over an ellipsoid $x^T A x \leqslant 1$ centered at the origin. Also consider the nonconvex extension with $B \notin \mathbb{S}^n_+$.*

Solution. Do this problem part by part.

(a) If $A \succ 0$, consider $y = A^{1/2} x$, and $\bar{c} = A^{-1/2} c$. The optimization problem becomes minimizing $\bar{c}^T y$ subject to $y^T y \leqslant 1$. This is minimizing a linear function over the unit ball, giving $y^* = -\bar{c}/\|\bar{c}\|_2$.

(b) Consider $y = A^{1/2}(x - x_c)$, $x = A^{-1/2} y + x_c$. The optimization problem becomes minimizing $c^T A^{-1/2} y + c^T x_c$ subject to $y^T y \leqslant 1$. This is another minimization question over the unit ball, so

$$y^* = -(1/\|A^{-1/2} c\|_2) A^{-1/2} c, \quad x^* = x_c - (1/\|A^{-1/2} c\|_2) A^{-1} c.$$

(c) If $B \succeq 0$, then the optimal value is zero. In the general case, consider

$$\lambda_{\min}(B) = \inf_{X^T x = 1} x^T B x.$$

To solve the optimization problem of minimizing $x^T B x$ subject to $x^T A x \leqslant 1$ with $a \succ 0$, consider $y = A^{1/2} x$. The problem becomes minimizing $y^T A^{-1/2} B A^{-1/2} y$ subject to $y^T y \leqslant 1$. Hence, the optimal value is $\lambda_{\min}(A^{-1/2} B A^{-1/2})$ if $y^T y = 1$.

Problem 24: Boyd 4.22

Consider the QCQP of minimizing $(1/2)x^T Px + q^T x + r$ subject to $x^T x \leqslant 1$, with $P \in \mathbb{S}^n_{++}$. Show that $x^* = -(P + \lambda I)^{-1}q$ where $\lambda = \max\left\{0, \overline{\lambda}\right\}$, and $\overline{\lambda}$ is the largest solution of the nonlinear equation

$$q^T(P + \lambda I)^{-2}q = 1.$$

Solution. x is optimal if and only if $x^T x < 1, Px + q = 0$ or $x^T x = 1, Px + q = -\lambda x$. Consider $Px = -q$. If the solution $\|P^{-1}q\|_2^2 \leqslant 1$, it is optimal. Otherwise, from the optimality conditions, x must satisfy $\|x\|_2 = 1$ and $(P + \lambda)x = -q$ for some $\lambda \geqslant 0$. Define $f(\lambda) = \|(P + \lambda)^{-1}q\|_2^2$. $f(0) = \|P^{-1}q\|_2^2 > 1$. f is monotonically decreasing and has limit zero as $\lambda \to \infty$. Therefore the nonlinear equation $f(\lambda) = 1$ has exactly one nonnegative solution $\overline{\lambda}$. Solving for λ gives

$$x^* = -(P + \overline{\lambda} I)^{-1}q.$$